

Generalizations of Chebyshev polynomials and Polynomial Mappings

Yang Chen[†] and James Griffin^{††}

Department of Mathematics, Imperial College,
180 Queen's Gate, London, SW7 2BZ, UK

Mourad E H Ismail^{†††}

Department of Mathematics, University of Central Florida,
Orlando, Florida, 32816, USA

Abstract

In this paper we show how polynomial mappings of degree \mathfrak{K} from a union of disjoint intervals onto $[-1, 1]$ generate a countable number of special cases of generalizations of Chebyshev polynomials. We also derive a new expression for these generalized Chebyshev polynomials for any genus g , from which the coefficients of x^n can be found explicitly in terms of the branch points and the recurrence coefficients. We find that this representation is useful for specializing to polynomial mapping cases for small \mathfrak{K} where we will have explicit expressions for the recurrence coefficients in terms of the branch points. We study in detail certain special cases of the polynomials for small degree mappings and prove a theorem concerning the location of the zeroes of the polynomials. We also derive an explicit expression for the discriminant for the genus 1 case of our Chebyshev polynomials that is valid for any configuration of the branch point.

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[†] e-mail: y.chen@ic.ac.uk

^{††} e-mail: j.c.griffin@ic.ac.uk

^{†††} e-mail :ismail@math.usf.edu

1 Introduction and Preliminaries.

Akhiezer [2], [1] and, Akhiezer and Tomčuk [3] introduced orthogonal polynomials on two intervals which generalize the Chebyshev polynomials. He observed that the study of properties of these polynomials requires the use of elliptic functions. In the case of more than two intervals, Tomčuk [17], investigated their Bernstein-Szegő asymptotics, with the theory of Hyperelliptic integrals, and found expressions in terms of a certain Abelian integral of the third kind. However, in his formulation, certain unknown points on a Hyperelliptic Riemann surface emerge due to the lack of an explicit representation of the original polynomials. This was

circumvented in [9]. In his book on elliptic functions [4] Akhiezer, obtained explicit formulas for the two interval case as an example of the application of elliptic functions to the theory of conformal mapping.

In 1984 Al-Salam, Allaway, and Askey introduced sieved ultraspherical polynomials which are orthogonal with respect to an absolutely continuous measure supported on $[-1, 1]$ but the weight function vanishes at $k + 1$ points. Ismail [12] observed that the vanishing of the weight function means that the polynomials are orthogonal on several adjacent intervals. He then introduced one additional parameter in the definition of the sieved ultraspherical polynomials which made them orthogonal on several intervals with gaps. In particular his polynomials include analogues of the Chebyshev polynomials of the first and second kind. Their continuous spectrum is $T_k^{-1}([-c, c])$, for $c \in (-1, 1)$, where T_k is a Chebyshev polynomial of the first kind. More over these polynomials have simple closed form expressions and elementary generating functions. More general sieved polynomials are in [7].

In [14] a study of Chebyshev type polynomials was carried out. They arose as extremal polynomials and are orthogonal on several intervals. The Chebyshev type polynomials satisfy similar extremal properties to the classical Chebyshev polynomials on $[-1, 1]$. In particular they deviate least from zero on a union of disjoint intervals. The extremal polynomials also have the property that they are orthogonal with respect to some weight function of the form,

$$\frac{\pi_{g-1}(x)}{\sqrt{\prod_{j=1}^{2g}(x - a_j)}}, \quad x \in E,$$

where $E := \cup_{j=1}^{2g-1}(a_j, a_{j+1})$ and the polynomial $\pi_{g-1}(x)$ is chosen such that the weight is positive on the interior of E . The existence of an extremal polynomials on any set E is clearly not guaranteed. It was shown in [14] that the existence of an extremal polynomial on a set E is equivalent to the asymptotic periodicity of the recurrence coefficients of a sequence of orthogonal polynomials on E . The extremal polynomial also has the property that it maps the set E onto $[-1, 1]$.

In [10], using a polynomial mapping of this kind, a prescription was given for the construction of a sequence of orthogonal polynomials on E from a sequence orthogonal on a single interval. The corresponding weight supported on E can be obtained using the following procedure, outlined in [10]. In this generality closed form expressions and generating functions are not available. This raises the question of finding a class of polynomials as general as possible which has the extremal properties of the polynomials in Peherstorfer [14] and the simple structural formulas of Ismail's polynomials in [12]. The first step was carried out in [9] and is further developed here. The present paper we develop this work further by borrowing ideas from polynomial mappings and sieved polynomials.

We start by choosing a polynomial of degree $\mathfrak{K} - 1$ and call this polynomial $W_{\mathfrak{K}-1}(t)$. Then we form the expression,

$$(1.1) \quad \frac{W_{\mathfrak{K}-1}(t)}{M_{\mathfrak{K}}(t) - x} = \frac{W_{\mathfrak{K}-1}(t)}{l_{\mathfrak{K}} \prod_{i=1}^{\mathfrak{K}} (t - M_i^{-1}(x))} = \sum_{i=1}^{\mathfrak{K}} \frac{\omega_i(x)}{t - M_i^{-1}(x)}$$

where, the $M_i^{-1}(x)$ are a complete set of inverse branches of $M_{\mathfrak{K}}(x)$ and $l_{\mathfrak{K}}$ is the leading coefficient of $M_{\mathfrak{K}}(t)$. Rearranging the above equation and taking the limit as $t \rightarrow M_i^{-1}(x)$ we find,

$$\omega_i(x) = \frac{W_{\mathfrak{K}-1}(M_i^{-1}(x))}{M'_{\mathfrak{K}}(M_i^{-1}(x))}.$$

By comparing coefficients of $t^{\mathfrak{K}-1}$ in (1.1) it is clear that,

$$\sum_{j=1}^{\mathfrak{K}} \omega_j(x) = c$$

where c is the quotient of the leading coefficients of $W_{\mathfrak{K}-1}(x)$ and $M_{\mathfrak{K}}(x)$. So, having chosen a particular polynomial $W_{\mathfrak{K}-1}(x)$ we are in a position to define the transformed weight function in the following way. Let E_0 be the interval $[a, b]$, and $P_n(x)$ the polynomials orthogonal with respect to $w_0(x)$ over E_0 :

$$\begin{aligned} h_n \delta_{nm} &= \int_a^b P_n(x) P_m(x) w_0(x) dx \\ &= \frac{1}{c} \int_a^b P_n(x) P_m(x) \left(\sum_{j=1}^{\mathfrak{K}} \omega_j(x) \right) w_0(x) dx \\ &= \frac{1}{c} \sum_{j=1}^{\mathfrak{K}} \int_{A_j} P_n(M_{\mathfrak{K}}(t)) P_m(M_{\mathfrak{K}}(t)) \frac{W_{\mathfrak{K}-1}(t)}{M'_{\mathfrak{K}}(t)} M'_{\mathfrak{K}}(t) w_0(M_{\mathfrak{K}}(t)) dt \\ &= \frac{1}{c} \int_E P_n(M_{\mathfrak{K}}(t)) P_m(M_{\mathfrak{K}}(t)) W_{\mathfrak{K}-1}(t) w_0(t) dt, \end{aligned}$$

where

$$E := \cup_{j=1}^{\mathfrak{K}} A_j,$$

and,

$$M_j^{-1} : [a, b] \rightarrow A_j.$$

So the new sequence of polynomials is orthogonal on the disjoint set E with respect to the weight function

$$\hat{w}(t) = W_{\mathfrak{K}-1}(t) w_0(M_{\mathfrak{K}}(t)).$$

It was shown in [10] that

$$\hat{P}_{n\mathfrak{K}}(x) = P_n(M_{\mathfrak{K}}(x)) \quad n = 0, 1, 2, \dots$$

This generates every \mathfrak{K}^{th} polynomial in the sequence. The polynomials $\hat{P}_{n\mathfrak{K}+j}(x)$ $1 \leq j \leq \mathfrak{K} - 1$, are referred to as the intermediate polynomials. Formulas relating the intermediate polynomials and recurrence coefficients of the new sequence to known quantities of the old sequence are given in [10].

Some explicit expressions for the Chebyshev polynomials on several intervals, along with many other important results can be found in [9]. This paper is organized as follows. In this section we recall the theta function representation of these polynomials which was derived in [9]. We also present some facts concerning the polynomials that will be referred to throughout this paper.

Section 2 is a study of two auxiliary polynomials $S_g(x; n)$ and $G_{g+1}(x; n)$, of degree g and $g + 1$ respectively. The auxiliary polynomials are fundamental to the theory of the generalization of Chebyshev

polynomials under consideration. We also derive for the first time non-linear difference equations satisfied by the recurrence coefficients that appear in the three term recurrence relation.

In section 3 we derive a product representation for our polynomials from which one can obtain the coefficients of the polynomial in terms of the recurrence coefficients and the branch points. This representation is a new one that involves the auxiliary polynomials studied in section 2. It is particularly useful when we consider certain configurations of the branch points where the recurrence coefficients simplify into an algebraic form.

In section 4 of this paper we show how certain polynomial mappings give rise to a countable number of special cases of our generalization of the Chebyshev polynomials. These cases correspond to when the recurrence coefficients are periodic. We give explicit formulas for the polynomials of degree not equal to an integer multiple of the period.

The monic form of the Chebyshev polynomials on several intervals will be denoted by $P_n(x)$. They are orthogonal on $E := [-1, \alpha] \cup_{j=1}^{g-1} [\beta_j, \alpha_{j+1}] \cup [\beta_g, 1]$ with respect to the following weight,

$$(1.2) \quad w(x) = \frac{1}{\pi} \sqrt{\frac{\prod_{j=1}^g (x - \alpha_j)}{(1 - x^2) \prod_{j=1}^g (x - \beta_j)}},$$

so that,

$$\int_E P_n(x) P_m(x) w(x) dx = \delta_{nm} h_n.$$

The polynomials of the second kind are

$$(1.3) \quad Q_n(x) = \int_E \frac{P_n(x) - P_n(t)}{x - t} w(t) dt.$$

Both $P_n(x)$ and $Q_n(x)$ satisfy the recurrence relation

$$(1.4) \quad x u_n(x) = u_{n+1}(x) + b_{n+1} u_n(x) + a_n u_{n-1}(x),$$

see [2]. The Stieltjes transform of the weight function is

$$\psi(z) := \int_E \frac{w(t) dt}{z - t} = \sqrt{\frac{\prod_{j=1}^g (z - \alpha_j)}{(z^2 - 1) \prod_{j=1}^g (z - \beta_j)}}, \quad z \notin E.$$

Note that,

$$\psi(x) = \frac{\pi}{i} w(x).$$

In [9], Chen and Lawrence considered a hyperelliptic Riemann surface of genus g ,

$$y^2(z) = (z^2 - 1) \prod_{j=1}^g (z - \alpha_j)(z - \beta_j),$$

from which a representation for the polynomials in terms of the associated Riemann theta function was constructed. They also gave a representation for the recurrence coefficients in terms of the theta functions. To formulate these representations the following facts regarding this Riemann surface are required.

A canonical basis of cycles is chosen on the surface that is composed of $2g$ closed contours. We denote by the a-cycles the g closed contours that lie on one sheet of the surface, and the b-cycles to be the remaining g contours that travel over both sheets. For a surface of genus g a basis of holomorphic differentials can be written in terms of g linearly independent differentials,

$$\left\{ \frac{dx}{y(x)}, \frac{x dx}{y(x)}, \dots, \frac{x^{g-1} dx}{y(x)} \right\}.$$

If we let,

$$A_{jk} = \int_{a_k} \frac{x^{g-j}}{y(x)} dx$$

then the $d\omega_j$ defined as,

$$d\omega_j = \sum_{k=1}^g (A^{-1})_{jk} \frac{x^{g-j}}{y(x)} dx,$$

form an orthonormal basis such that,

$$\int_{a_j} d\omega_k = \delta_{jk}.$$

We can then define the period matrix B whose entries are,

$$B_{jk} = \int_{b_j} d\omega_k.$$

The Riemann theta function associated with the surface is defined to be,

$$\vartheta(s; B) = \vartheta(s) := \sum_{t \in \mathbb{Z}^g} \exp(i\pi(t, Bt) + 2\pi i(t, s)),$$

where s is a g -vector, and (u, v) denotes the standard scalar product of two vectors. Convergence of the series is assured since $\Im B$ is a positive definite matrix. The theta function has the following properties. For $t \in \mathbb{Z}^g$,

$$\begin{aligned} \vartheta(-s) &= \vartheta(s) \\ \vartheta(s + t) &= \vartheta(s) \\ \vartheta(s + Bt) &= e^{-i\pi[(t, Bt) + 2(t, s)]} \vartheta(s) \end{aligned}$$

Both the polynomials and the recurrence coefficients are given below. The expressions are taken from [9]. However, in view of lemma 10.2 in [9] and the quasi-periodicity of the theta functions we have simplified the expressions slightly.

$$(1.5) \quad P_n(x) = \frac{1}{2} \left(\mathcal{E}_n(\mathfrak{p}_x) + \tilde{\mathcal{E}}_n(\mathfrak{p}_x) \right),$$

where, for $n \geq 1$,

$$\mathcal{E}_n(\mathbf{p}_x) = 2e^{-n(\chi_0 + \Omega(\mathbf{p}_x))} \frac{\vartheta(\mathbf{u}^+) \vartheta(\mathbf{u}_x + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}}) \vartheta(\mathbf{u}_x)}$$

and,

$$\tilde{\mathcal{E}}_n(\mathbf{p}_x) = 2e^{n(\Omega(\mathbf{p}_x) - \chi_0)} \frac{\vartheta(\mathbf{u}^+) \vartheta(\mathbf{u}_x - n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}}) \vartheta(\mathbf{u}_x)}.$$

Using this notation,

$$(1.6) \quad Q_n(x) = \frac{\psi(x)}{2} \left(\tilde{\mathcal{E}}_n(\mathbf{p}_x) - \mathcal{E}_n(\mathbf{p}_x) \right).$$

Here, \mathbf{p}_x is a point on the top sheet of the Riemann surface that corresponds to a point x in the complex plane and,

$$\begin{aligned} \Omega(\mathbf{p}_x) &= \int_{\mathbf{p}_1}^{\mathbf{p}_x} d\Omega \\ &:= \int_{\mathbf{p}_1}^{\mathbf{p}_x} \frac{t^g + \sum_{j=0}^{g-1} k_j t^j}{y(t)} dt. \end{aligned}$$

The k_j can be determined from the condition that,

$$\int_{a_j} d\Omega = 0, \quad j = 1, \dots, g.$$

\mathbf{u}_x is defined to be a g-vector,

$$\mathbf{u}_x := \left(\int_{p_1}^{p_x} d\omega_1, \dots, \int_{p_1}^{p_x} d\omega_g \right)^T.$$

and $\mathbf{u}^\pm := \mathbf{u}_{\infty^\pm}$. If $\hat{\mathbf{B}}$ is the vector whose components are,

$$\hat{\mathbf{B}}_j = \frac{1}{2\pi i} \int_{b_j} d\Omega, \quad j = 1, \dots, g.$$

then,

$$\hat{\mathbf{B}} \equiv \mathbf{u}^+ - \mathbf{u}^- \pmod{\mathbb{Z}^g}.$$

Finally, the equilibrium potential of the set E is

$$(1.7) \quad \chi_0 = \int_{\mathbf{p}_1}^{\infty^+} \left(\frac{t^g + \sum_{j=0}^g k_j t^j}{y(t)} - \frac{1}{t} \right) dt,$$

With this notation, we have the following expressions for the recurrence coefficients,

$$(1.8) \quad a_n = \begin{cases} 2e^{-2\chi_0} \frac{\vartheta(\mathbf{u}^+ + \hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - \hat{\mathbf{B}})} & \text{if } n = 1 \\ e^{-2\chi_0} \frac{\vartheta(\mathbf{u}^+ - (n-1)\hat{\mathbf{B}}) \vartheta(\mathbf{u}^+ + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ + (n-1)\hat{\mathbf{B}}) \vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}})} & \text{if } n > 1 \end{cases}$$

and,

$$(1.9) \quad b_n = \frac{1}{2} \sum_{j=1}^g (\beta_j - \alpha_j) + \sum_{j=1}^g (A^{-1})_{j1} \left[\frac{\vartheta'_j(\mathbf{u}^+ - \hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - \hat{\mathbf{B}})} - \frac{\vartheta'_j(\mathbf{u}^+)}{\vartheta(\mathbf{u}^+)} + \frac{\vartheta'_j(\mathbf{u}^+ - (n-1)\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - (n-1)\hat{\mathbf{B}})} - \frac{\vartheta'_j(\mathbf{u}^+ - n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}})} \right].$$

Here,

$$\vartheta'_j(\mathbf{u}) := \frac{\partial}{\partial u_j} \vartheta(\mathbf{u})$$

Also,

$$(1.10) \quad h_n = 2e^{-2n\chi_0} \frac{\vartheta(\mathbf{u}^+ + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}})}.$$

Theorem 1.1 *The polynomials of the second kind $Q_n(x)$ satisfy the following orthogonality relation,*

$$(1.11) \quad \int_E Q_n(x) Q_m(x) \frac{1}{w(x)} dx = \delta_{nm} \tilde{h}_n,$$

where,

$$\tilde{h}_n = \pi^2 h_n.$$

Proof : We wish to evaluate the following integral,

$$I_j = \int_E \frac{Q_n(t)}{w(t)} t^j dt \quad 0 \leq j < n-1.$$

We have,

$$\begin{aligned} Q_n(z) &= P_n(z) \psi(z) - \int_E \frac{P_n(t)}{z-t} w(t) dt \\ &= P_n(z) \psi(z) - \frac{1}{z} \int_E \frac{P_n(t)}{1-\frac{t}{z}} w(t) dt \\ &= P_n(z) \psi(z) - \frac{1}{z} \int_E P_n(t) \sum_{j=0}^{\infty} \left(\frac{t}{z} \right)^j dt \\ &= P_n(z) \psi(z) - \left(\frac{h_n}{z^{n+1}} + O\left(\frac{1}{z^{n+2}} \right) \right) \end{aligned}$$

where the last line follows from the orthogonality of the $P_n(x)$. Therefore,

$$\frac{Q_n(z)}{\psi(z)} = P_n(z) - \frac{1}{\psi(z)} \left(\frac{h_n}{z^{n+1}} + O\left(\frac{1}{z^{n+2}} \right) \right).$$

We rewrite I_j as a contour integral using a closed contour Λ [9] [Fig. 1], in the slit complex plane that encircles all of the branch points. By continuous deformation of the contour onto the intervals that make up E we have,

$$\begin{aligned} I_j &= \int_E \frac{Q_n(t)}{w(t)} t^j dt \\ &= \frac{\pi}{2i} \int_{\Lambda} \frac{Q_n(z)}{\psi(z)} z^j dz. \end{aligned}$$

So I_j can be written as,

$$I_j = \int_{\Lambda} P_n(z) z^j dz - \int_{\Lambda} \frac{\frac{h_n}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right)}{\psi(z)} z^j dz.$$

The differential $P_n(z) z^j dz$ has no residue at infinity for any j . The differential $\frac{z^j}{z^{n+1}\psi(z)} dz$ has no residue at infinity for $j = 0, \dots, n-2$. Consequently, for these values of j , both integrals are zero which proves the orthogonality of the $Q_n(z)$ with respect to $\frac{1}{w(z)}$. The integral I_{n-1} is equal to the square of the weighted L_2 norm of $Q_n(z)$. In this case there is a residue at infinity. The residue is $-h_n$ and therefore,

$$\tilde{h}_n = \pi^2 h_n$$

□

Theorem 1.2 *Let \tilde{E} be the reflection of E around the point $x = 0$, namely*

$$\tilde{E} := [-1, -\beta_g] \cup_{j=g-1}^1 [-\alpha_{j+1}, -\beta_j] \cup [-\alpha_1, 1].$$

Let $\{\tilde{P}_n(x)\}$ be the monic polynomials orthogonal on \tilde{E} with respect to $\tilde{w}(x)$, where,

$$\tilde{w}(x) = \frac{1}{\pi} \sqrt{\frac{\prod_{j=1}^g (x + \beta_j)}{(1-x^2) \prod_{j=1}^g (x + \alpha_j)}}.$$

Then we have for $n \geq g$

$$(1.12) \quad \tilde{P}_n(x) = \frac{(-1)^{n+g}}{D_n \prod_{j=1}^g (x + \beta_j)} \begin{vmatrix} P_{n-g}(\beta_1) & \dots & \dots & \dots & P_{n+g}(\beta_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n-g}(\beta_g) & \dots & \dots & \dots & P_{n+g}(\beta_g) \\ Q_{n-g}(\alpha_1) & \dots & \dots & \dots & Q_{n+g}(\alpha_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{n-g}(\alpha_g) & \dots & \dots & \dots & Q_{n+g}(\alpha_g) \\ P_{n-g}(-x) & \dots & P_n(-x) & \dots & P_{n+g}(-x) \end{vmatrix}.$$

If $n < g$ then,

$$\tilde{P}_n(x) = \frac{(-1)^{n+g}}{D_n \prod_{j=1}^g (x + \beta_j)} \begin{vmatrix} 0 & \dots & \dots & 0 & P_0(\beta_1) & \dots & P_{n+g}(\beta_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & P_0(\beta_g) & \dots & P_{n+g}(\beta_g) \\ \alpha_1^0 & \alpha_1 & \dots & \alpha_1^{g-n-1} & Q_0(\alpha_1) & \dots & Q_{n+g}(\alpha_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_g^0 & \alpha_g & \dots & \alpha_g^{g-n-1} & Q_0(\alpha_g) & \dots & Q_{n+g}(\alpha_g) \\ 0 & 0 & 0 & 0 & P_0(-x) & \dots & P_{n+g}(-x) \end{vmatrix}.$$

In both equations, D_n is equal to the coefficient of $P_{n+g}(-x)$ appearing in each determinant.

Proof: Starting with the orthogonality relation of the $\tilde{P}_n(x)$ we make the change of variables $x \rightarrow -x$.

$$\begin{aligned} \tilde{h}_n \delta_{nm} &= \int_{\tilde{E}} \tilde{P}_n(x) \tilde{P}_m(x) \sqrt{\frac{\prod_{j=1}^g (x + \beta_j)}{(1-x^2) \prod_{j=1}^g (x + \alpha_j)}} dx \\ &= \int_E \tilde{P}_n(-x) \tilde{P}_m(-x) \sqrt{\frac{\prod_{j=1}^g (x - \beta_j)}{(1-x^2) \prod_{j=1}^g (x - \alpha_j)}} dx \\ &= \int_E \tilde{P}_n(-x) \tilde{P}_m(-x) \sqrt{\frac{\prod_{j=1}^g (x - \alpha_j)}{(1-x^2) \prod_{j=1}^g (x - \beta_j)}} \left(\frac{\prod_{j=1}^g (x - \beta_j)}{\prod_{j=1}^g (x - \alpha_j)} \right) dx \end{aligned}$$

So the $\tilde{P}_n(-x)$ are orthogonal with respect to a weight function that is $w(x)$ multiplied by a rational function. In general if we know the polynomials orthogonal with respect to a weight $\mu(x)$ and we wish to find the polynomials orthogonal with respect to $R(x)\mu(x)$ where $R(x)$ is a rational function that is positive on the interval of orthogonality we may invoke a theorem of Uvarov that gives these polynomials in terms of the original ones. The theorem was first published in [18] and also appears in [19]. In our case the rational function has zeros and poles at the end points of the subintervals that make up E . However, a brief examination of the proof will show that the theorem is still applicable in this case. The result follows directly from the theorem. \square

In section 2 we wish to study the following polynomials,

$$(1.13) \quad S_g(x; n) := P_n^2(x) \prod_{j=1}^g (x - \alpha_j) - Q_n^2(x)(x^2 - 1) \prod_{j=1}^g (x - \beta_j)$$

and,

$$(1.14) \quad G_{g+1}(x; n) := P_n(x)P_{n-1}(x) \prod_{j=1}^g (x - \alpha_j) - Q_n(x)Q_{n-1}(x)(x^2 - 1) \prod_{j=1}^g (x - \beta_j).$$

These functions are fundamental in the theory of the generalizations of Chebyshev polynomials considered here.

Theorem 1.3 *The functions $S_g(x; n)$ and $G_{g+1}(x; n)$ are polynomials of degree g and $g + 1$ respectively.*

Proof: From (1.3) we know that,

$$Q_n(z) = P_n(z)\psi(z) - \int_E \frac{P_n(t)}{z-t} w(t) dt.$$

Expanding the integrand in powers of t we have,

$$\frac{1}{z-t} = \frac{1}{z} + \frac{t}{z^2} + \frac{t^2}{z^3} + \dots$$

Therefore, as $z \rightarrow \infty^+$, we have

$$P_n(z)\psi(z) - Q_n(z) = O\left(\frac{1}{z^{n+1}}\right),$$

and similarly,

$$P_n(z)\psi(z) + Q_n(z) = O(z^{n-1}).$$

Multiplying the two together gives

$$P_n^2(z)\psi^2(z) - Q_n^2(z) = O\left(\frac{1}{z^2}\right).$$

Therefore,

$$\prod_{j=1}^g (z - \alpha_j) P_n^2(z) - (z^2 - 1) \prod_{j=1}^g (z - \beta_j) Q_n^2(z) = O(z^g).$$

Since the left hand side is a polynomial it follows that it must be of degree g . For (1.14) we form the following product,

$$\begin{aligned} & (P_n(z)\psi(z) - Q_n(z))(P_{n-1}(z)\psi(z) + Q_{n-1}(z)) \\ &= P_n(z)P_{n-1}(z)\psi(z) - Q_n(z)Q_{n-1}(z) - \psi(z)h_{n-1} \end{aligned}$$

where we have used,

$$P_{n-1}(z)Q_n(z) - P_n(z)Q_{n-1}(z) = h_{n-1},$$

which is the Wronskian. As a result,

$$\prod_{j=1}^g (z - \alpha_j) P_n(z)P_{n-1}(z) - (z^2 - 1) \prod_{j=1}^g (z - \beta_j) Q_n(z)Q_{n-1}(z) = O(z^{g+1}).$$

□

From (1.13) and (1.14) we have,

$$(1.15) \quad \mathcal{E}_n(\mathbf{p}_x) \tilde{\mathcal{E}}_n(\mathbf{p}_x) = \frac{S_g(x; n)}{\prod_{j=1}^g (x - \alpha_j)}.$$

Also,

$$(1.16) \quad \mathcal{E}_n(\mathbf{p}_x) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_x) = \frac{G_{g+1}(x; n) + h_{n-1}y(x)}{\prod_{j=1}^g (x - \alpha_j)}$$

$$(1.17) \quad \tilde{\mathcal{E}}_n(\mathbf{p}_x) \mathcal{E}_{n-1}(\mathbf{p}_x) = \frac{G_{g+1}(x; n) - h_{n-1}y(x)}{\prod_{j=1}^g (x - \alpha_j)}.$$

An alternative representation for the polynomials of the first and second kind was also given in [9] resulting from a coupled system of differential equations.

$$(1.18) \quad P_n(z) := \rho_n(z) \cos \Psi_n(z)$$

and,

$$(1.19) \quad Q_n(z) := i\psi(z)\rho_n(z) \sin \Psi_n(z),$$

where,

$$\rho_n(z) = \sqrt{\frac{S_g(z; n)}{\prod_{j=1}^g (z - \alpha_j)}}$$

and,

$$\Psi_n(z) = i \int_1^z \frac{(\sum_{j=0}^{g-1} c_j(n)t^j - nt^g + \frac{1}{2} \sum_{j=1}^g \frac{y_j(n)}{t - \gamma_j(n)})}{y} dt.$$

where the branch of the square root is chosen in such a way that $\rho_n(z) \rightarrow \sqrt{2h_n}$, as $\Re z \rightarrow \infty$ and $2h_n$ is the leading coefficient of $S_g(z; n)$, to be shown later. Here the path of integration is from 1 to an arbitrary complex point z with the property $\Im z > 0$, and the path of integration is entirely in the upper half plane. Expressions for the real values are then obtained by the analytical continuation of those above allowing $z \rightarrow x \in \mathbb{R}$.

In these expressions, the $\gamma_j(n)$ are the zeroes of $S_g(x, n)$, and $y_i = y(\mathbf{p}_x)|_{\mathbf{p}_x = \gamma_j(n)}$. The $c_j(n)$ are the first $g + 1$ expansion coefficients of $\frac{d}{dx} \ln \mathcal{E}_n(\mathbf{p}_x)$ in terms of the following basis,

$$\left\{ \frac{1}{y}, \frac{x}{y}, \dots, \frac{x^g}{y}, \frac{1}{x - \alpha_1}, \dots, \frac{1}{x - \alpha_g}, \frac{y + y_1}{y(x - \gamma_1(n))}, \dots, \frac{y + y_g}{y(x - \gamma_g(n))} \right\}$$

This is the basis for meromorphic functions on the Riemann surface defined by $y(x)$ whose zeroes must include simple zeroes at $\pm\infty$ and whose poles must be from amongst the set $\mathbf{p}_1, \mathbf{p}_{-1}, \mathbf{p}_{\alpha_j}, \mathbf{p}_{\beta_j}, \mathbf{p}_{\gamma_j} : j = 1, \dots, g$, where the poles at the α points can be at most double poles and the other poles must be simple poles.

2 Evaluating the polynomials $S_g(x; n)$ and $G_{g+1}(x; n)$

In this section we give an algorithm for determining the coefficients of $S_g(x; n)$ and $G_{g+1}(x; n)$ in terms of the recurrence coefficients and the branch points. Let

$$S_g(x; n) = \sum_{j=0}^g \eta_j(n) x^j$$

and,

$$G_{g+1}(x; n) = \sum_{j=0}^{g+1} \xi_j(n) x^j.$$

Then we have,

$$(2.1) \quad \psi(z) P_n^2(z) - \frac{Q_n^2(z)}{\psi(z)} = \frac{\eta_g(n) z^g + \dots + \eta_0(n)}{y(z)}$$

$$(2.2) \quad \psi(z) P_n(z) P_{n-1}(z) - \frac{Q_n(z) Q_{n-1}(z)}{\psi(z)} = \frac{\xi_{g+1}(n) z^{g+1} + \dots + \xi_0(n)}{y(z)},$$

To find the $\eta_j(n)$ coefficients we integrate both sides of (2.1) along the contour Λ . Note that,

$$\psi(z) = \frac{\pi}{i} W(z),$$

where

$$W(z) = \frac{i}{\pi} \sqrt{\frac{\prod_{j=1}^g (z - \alpha_j)}{(z^2 - 1) \prod_{j=1}^g (z - \beta_j)}},$$

and for $t \in E$,

$$(2.3) \quad \lim_{\epsilon \rightarrow 0+} W(t + i\epsilon) = w(t) := \frac{1}{\pi} \sqrt{\frac{\prod_{j=1}^g (t - \alpha_j)}{(1 - t^2) \prod_{j=1}^g (t - \beta_j)}}.$$

We obtain the following,

$$(2.4) \quad \frac{\pi}{i} \int_{\Lambda} W(z) P_n^2(z) dz - \frac{i}{\pi} \int_{\Lambda} \frac{Q_n^2(z)}{W(z)} dz = \sum_{j=0}^g \eta_j(n) I_j,$$

where,

$$I_j = \int_{\Lambda} \frac{z^j}{y(z)} dz.$$

The I_j can be evaluated by evaluating the residue at infinity. in which case,

$$I_j = 0 \quad j < g$$

and,

$$I_{g+k} = \int_{\Lambda} \frac{z^{g+k}}{\left[(z^2 - 1) \prod_{j=1}^g (z - \alpha_j)(z - \beta_j)\right]^{1/2}} dz$$

which under the substitution $z = 1/Z$ becomes,

$$I_{g+k} = - \int_{\Omega} \frac{dZ}{Z^{k+1} \left[(1 - Z^2) \prod_{j=1}^g (1 - \alpha_j Z)(1 - \beta_j Z)\right]^{1/2}}.$$

So in general the integrand of I_{g+k} has a pole of order $k + 1$ at infinity. Hence if,

$$f(Z) = \left[(1 - Z^2) \prod_{j=1}^g (1 - \alpha_j Z)(1 - \beta_j Z) \right]^{-1/2}$$

then,

$$(2.5) \quad I_{g+k} = -2\pi i \frac{f^{(k)}(0)}{k!}$$

where $f^{(k)}(Z)$ denotes the k th derivative of f . Deforming Λ onto E (2.4) becomes,

$$4h_n = \frac{i}{\pi} \eta_g(n) I_g.$$

From (2.5) we have,

$$I_g = -2\pi i,$$

and therefore,

$$\eta_g(n) = 2h_n.$$

To find the remaining $\eta_j(n)$ coefficients we multiply (2.1) by successive powers of z and perform the same integration. This leaves us with the following system of equations that we can use to solve for the $\eta_g(n)$,

$$\begin{aligned} r_0(n) &= I_g \eta_g(n) \\ r_1(n) &= I_{g+1} \eta_g(n) + I_g \eta_{g-1}(n) \\ r_2(n) &= I_{g+2} \eta_g(n) + I_{g+1} \eta_{g-1}(n) + I_g \eta_{g-2}(n) \\ &\vdots \\ r_g(n) &= \sum_{j=0}^g I_{g+j} \eta_j. \end{aligned}$$

Here,

$$(2.6) \quad r_k(n) = \frac{4\pi}{i} \int_E z^k P_n^2(z) w(z) dz.$$

These constants are obtained by iterating the three term recurrence relation k times in (2.6). The recurrence relation can also be expressed in matrix form by,

$$Lp = zp,$$

where,

$$L = \begin{pmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & b_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_3 & b_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_4 & b_5 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \end{pmatrix} \quad p = \begin{pmatrix} P_0(z) \\ P_1(z) \\ P_2(z) \\ P_3(z) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

In this case,

$$\begin{aligned} z^k p &= L^k p \\ zP_n(z) &= \sum_{m=0}^{\infty} L_{nm} p_m \\ z^k P_n(z) &= \sum_{m=0}^{\infty} [L^k]_{nm} P_m(z). \end{aligned}$$

From the expression for $r_k(n)$ in (2.6) it is clear that,

$$(2.7) \quad r_k(n) = \frac{4\pi h_n}{i} [L^k]_{n-1,n}.$$

To find the $\xi_j(n)$ we use the same method. We know from the proof of theorem (1.3) that,

$$\xi_{g+1}(n) = h_{n-1}.$$

Multiplying (2.2) by successive powers of z and integrating we have the following system of equations,

$$\begin{aligned} 0 &= I_{g+1}\xi_{g+1}(n) + I_g\xi_g(n) \\ s_1(n) &= I_{g+2}\xi_{g+1}(n) + I_{g+1}\xi_g(n) + I_g\xi_{g-1}(n) \\ s_2(n) &= I_{g+3}\xi_{g+1}(n) + I_{g+2}\xi_g(n) + I_{g+1}\xi_{g-1}(n) + I_g\xi_{g-2}(n) \\ &\vdots \\ s_g(n) &= \sum_{j=0}^{g+1} I_{g+j}\xi_j(n). \end{aligned}$$

Here,

$$s_k(n) = \frac{4\pi}{i} \int_E z^k P_n(z) P_{n-1}(z) w(z) dz.$$

In terms of the Jacobi matrix,

$$s_k(n) = \frac{4\pi h_n}{i} [L^k]_{nn}.$$

For $g = 1$ we have,

$$\begin{aligned} S_1(x, n) &= 2h_n \left(x + b_{n+1} - \frac{(\alpha + \beta)}{2} \right) \quad n \geq 1 \\ G_2(x, n) &= h_{n-1} \left(x^2 - \frac{\alpha + \beta}{2} x + 2a_n - \frac{1}{8}(\alpha - \beta)^2 - \frac{1}{2} \right). \quad n \geq 2 \end{aligned}$$

$$\begin{aligned} S_1(x, 0) &= (x - \alpha) \\ G_2(x, 1) &= (x - b_1)(x - \alpha). \end{aligned}$$

The auxiliary polynomials for $g = 2$ can be found in appendix B.

Non-linear difference equations

We can use the polynomials studied in this section to derive a pair of non linear difference equations for each genus that are satisfied by the recurrence coefficients $\{a_n\}$ and $\{b_n\}$. Using the three term recurrence relation,

$$\begin{aligned} \frac{S_g(z; n)}{\prod_{j=1}^g (z - \alpha_j)} &= \mathcal{E}_n(\mathbf{p}_z) \tilde{\mathcal{E}}_n(\mathbf{p}_z) \\ &= (z - b_n)^2 \mathcal{E}_{n-1}(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) \\ &\quad - a_{n-1}(z - b_n)(\mathcal{E}_{n-2}(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) + \tilde{\mathcal{E}}_{n-2}(\mathbf{p}_z) \mathcal{E}_{n-1}(\mathbf{p}_z)) + a_{n-1}^2 \mathcal{E}_{n-2}(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-2}(\mathbf{p}_z) \\ (2.8) \quad &= \frac{(z - b_n)^2 S_g(z; n-1) + a_{n-2}^2 S_g(z; n-2) - 2a_{n-1}(z - b_n) G_{g+1}(z; n-1)}{\prod_{j=1}^g (z - \alpha_j)}. \end{aligned}$$

Evaluating both sides at $z = b_n$ gives,

$$S_g(b_n; n) = a_{n-1}^2 S_g(b_n, n-2).$$

If we write this out fully for $g = 1$ we have,

$$(2.9) \quad a_n \left(b_{n+1} + b_n - \frac{\alpha + \beta}{2} \right) = a_{n-1} \left(b_n + b_{n-1} - \frac{\alpha + \beta}{2} \right).$$

This is our first difference relation for the recurrence coefficients. To get the second we note that,

$$\begin{aligned} \frac{G_{g+1}(z; n)}{\prod_{j=1}^g (z - \alpha_j)} &= \frac{1}{2} \left(\mathcal{E}_n(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) + \tilde{\mathcal{E}}_n(\mathbf{p}_z) \mathcal{E}_{n-1}(\mathbf{p}_z) \right) \\ &= \frac{1}{2} \left(((z - b_n) \mathcal{E}_{n-1}(\mathbf{p}_z) - a_{n-1} \mathcal{E}_{n-2}(\mathbf{p}_z)) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) \right. \\ &\quad \left. + (z - b_n) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) - a_{n-1} \tilde{\mathcal{E}}_{n-2}(\mathbf{p}_z)) \mathcal{E}_{n-1}(\mathbf{p}_z) \right) \\ &= (z - b_n) \mathcal{E}_{n-1}(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) - \frac{a_{n-1}}{2} \left(\mathcal{E}_{n-2}(\mathbf{p}_z) \tilde{\mathcal{E}}_{n-1}(\mathbf{p}_z) + \tilde{\mathcal{E}}_{n-2}(\mathbf{p}_z) \mathcal{E}_{n-1}(\mathbf{p}_z) \right) \\ (2.10) \quad &= \frac{(z - b_n) S_g(z; n-1) - a_{n-1} G_{g+1}(z; n-1)}{\prod_{j=1}^g (z - \alpha_j)}. \end{aligned}$$

If we evaluate this at $z = b_n$ we get ,

$$G_{g+1}(b_n; n) = -a_{n-1}G_{g+1}(b_n; n-1).$$

Writing this out for genus 1,

$$(2.11) \quad a_n + a_{n-1} = \frac{1}{2} + \frac{(\beta - \alpha)^2}{8} + \frac{\alpha + \beta}{2}b_n - b_n^2.$$

This is our second difference equation for the recurrence coefficients. Both equations are valid for $n \geq 3$.

3 Product representation for $P_n(x)$ and $Q_n(x)$

In this section we use the polynomials derived in section 2 to derive a representation for the generalized Chebyshev polynomials where the coefficients of the powers of x are given in terms of the branch points and the recurrence coefficients. From (1.5) we have

$$P_n(x) = \frac{1}{2} \left(\mathcal{E}_n(\mathbf{p}_x) + \tilde{\mathcal{E}}_n(\mathbf{p}_x) \right).$$

We define the following functions,

$$f_+(x; n) := \frac{\mathcal{E}_n(\mathbf{p}_x)}{\mathcal{E}_{n-1}(\mathbf{p}_x)}$$

$$f_-(x; n) := \frac{\tilde{\mathcal{E}}_n(\mathbf{p}_x)}{\tilde{\mathcal{E}}_{n-1}(\mathbf{p}_x)}.$$

Using (1.16) and (1.17) we can write,

$$(3.1) \quad f_+(x; n) = \frac{\mathcal{E}_n(\mathbf{p}_x)\tilde{\mathcal{E}}_{n-1}(\mathbf{p}_x)}{\mathcal{E}_{n-1}(\mathbf{p}_x)\tilde{\mathcal{E}}_n(\mathbf{p}_x)} = \frac{G_{g+1}(x; n) + h_{n-1}y(x)}{S_g(x; n-1)}.$$

Similarly,

$$(3.2) \quad f_-(x; n) = \frac{\tilde{\mathcal{E}}_n(\mathbf{p}_x)\mathcal{E}_{n-1}(\mathbf{p}_x)}{\tilde{\mathcal{E}}_{n-1}(\mathbf{p}_x)\mathcal{E}_n(\mathbf{p}_x)} = \frac{G_{g+1}(x; n) - h_{n-1}y(x)}{S_g(x; n-1)}.$$

Since, $\mathcal{E}_0(\mathbf{p}_x) = 1$, we have, for any $x \in \mathbb{C}$

$$(3.3) \quad P_n(x) = \frac{1}{2} \left(\prod_{j=1}^n f_+(x, j) + \prod_{j=1}^n f_-(x, j) \right).$$

Similarly,

$$(3.4) \quad Q_n(x) = \frac{\psi(x)}{2} \left(\prod_{j=1}^n f_+(x, j) - \prod_{j=1}^n f_-(x, j) \right).$$

(3.3) and (3.4) are valid for any genus g and require only knowledge of the branch points and the recurrence coefficients. They do not require any knowledge of the γ points. If we look at the genus 1 case we find that,

$$f_{\pm}(x; n) = \frac{x^2 - \frac{\alpha+\beta}{2}x + 2a_n - \frac{1}{8}(\alpha - \beta)^2 - \frac{1}{2} \pm \sqrt{(x^2 - 1)(x - \alpha)(x - \beta)}}{2(x + b_{n+1} - \frac{\alpha+\beta}{2})}.$$

If we take the limit as $\alpha \rightarrow \beta$ we see that,

$$S_1(x, n) \rightarrow 2h_n(x - \alpha) \quad n \geq 1$$

$$G_2(x, n) \rightarrow h_{n-1}x(x - \alpha),$$

and,

$$f_{\pm}(x, n) \rightarrow \frac{1}{2}(x \pm \sqrt{x^2 - 1}) = \frac{1}{2}e^{\pm i\theta} \quad x = \cos \theta \quad n \geq 2.$$

Note that,

$$f_{\pm}(x, 1) \rightarrow e^{\pm i\theta}.$$

Thus we can see that (3.3) is a natural generalization of the following formula for the classical Chebyshev polynomials,

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2^n}.$$

In general, (3.3) and (3.4) are very effective representations of the polynomials for the purpose of extracting the polynomial coefficients. The appearance of the theta function of the Riemann surface is still apparent in the expression for the recurrence coefficients. One may also wish to study certain special cases of the Akhiezer polynomials when the recurrence coefficients become algebraic functions of the branch points α_i and β_i . In this case (3.1) is particularly useful as one could insert directly the value of the recurrence coefficients. In the next section we will encounter some of these special cases. The following formulas follow naturally from (3.1):

$$(3.5) \quad P_n(x) = \frac{G_{g+1}(x, n)}{S_g(x, n-1)}P_{n-1}(x) + \frac{h_{n-1}(x^2 - 1) \prod_{j=1}^g (x - \beta_j)}{S_g(x, n-1)}Q_{n-1}(x),$$

and the corresponding formula for the Q_n is,

$$(3.6) \quad Q_n(x) = \frac{G_{g+1}(x, n)}{S_g(x, n-1)}Q_{n-1}(x) + \frac{h_{n-1} \prod_{j=1}^g (x - \alpha_j)}{S_g(x, n-1)}P_{n-1}(x).$$

Letting $\alpha_i \rightarrow \beta_i$ in (3.3) we obtain as expected the analogous formula for the Chebyshev case:

$$2T_n(x) = xT_{n-1}(x) + (x^2 - 1)U_{n-1}(x).$$

Here $T_n(x)$ and $U_n(x)$ are monic Chebyshev polynomials of the first and second kind respectively.

Now we show how the product representations of (3.1) and (3.2) can be used to re-express the differential relations derived in [9], where $P_n(x)$ and the $Q_n(x)$ were shown to satisfy,

$$(3.7) \quad P'_n(x) = f_1(x; n)P_n(x) + f_2(x; n)Q_n(x),$$

and

$$(3.8) \quad Q'_n(x) = f_3(x; n)P_n(x) + f_4(x; n)Q_n(x),$$

where,

$$\begin{aligned} f_1(x; n) &= \frac{1}{2} \sum_{j=1}^g \left(\frac{1}{x - \gamma_j} - \frac{1}{x - \alpha_j} \right) \\ f_2(x; n) &= \frac{nx^g - \sum_{j=0}^{g-1} c_j x^j - \frac{1}{2} \sum_{j=1}^g \frac{y_j}{x - \gamma_j}}{\prod_{j=1}^g (x - \alpha_j)}, \end{aligned}$$

and,

$$f_3(x; n) = \psi^2(x) f_2(x)$$

$$f_4(x; n) = \frac{\psi'(x)}{\psi(x)} + f_1(x).$$

The definitions of all the constants appearing in the above formulas can be found in section 1. From [9],

$$f_1(x; n) = \frac{\rho'_n(x)}{\rho_n(x)},$$

and,

$$(3.9) \quad f_2(x; n) = i \frac{\Psi'_n(x)}{\psi(x)}.$$

Using (3.1) we can express $f_1(x; n)$ and $f_2(x; n)$ in more explicit terms. Recall that,

$$\rho_n(x) = \sqrt{\frac{S_g(x, n)}{\prod_{j=1}^g (x - \alpha_j)}}.$$

Therefore, we can write,

$$(3.10) \quad f_1(x; n) = \frac{\rho'_n(x)}{\rho_n(x)} = \frac{1}{2} \left(\frac{S'_g(x, n)}{S_g(x, n)} - \sum_{j=1}^g \frac{1}{x - \alpha_j} \right).$$

This is simply a restatement of the earlier definition if we recall that the $\gamma_i(n)$ are the zeroes of $S_g(x, n)$. In order to re-express $f_2(x; n)$, we observe that

$$\prod_{j=1}^n f_{\pm}(x, j) = \rho_n(x) e^{\pm i \Psi_n(x)}.$$

Therefore,

$$\prod_{j=1}^n \frac{f_+(x, j)}{f_-(x, j)} = e^{2i \Psi_n(x)} \quad \text{or} \quad \ln \prod_{j=1}^n \frac{f_+(x, j)}{f_-(x, j)} = 2i \Psi_n(x) \pmod{2\pi}.$$

From the definitions of $f_{\pm}(x, j)$ we have,

$$(3.11) \quad 2i \Psi_n(x) = \sum_{j=1}^n \ln \left(\frac{G_{g+1}(x; j) + h_{j-1} y(x)}{G_{g+1}(x; j) - h_{j-1} y(x)} \right) \pmod{2\pi}.$$

Hence,

$$(3.12) \quad i \Psi'_n(x) = \sum_{j=1}^n \left[\frac{y'(x) G_{g+1}(x; j) - y(x) G'_{g+1}(x; j)}{S_g(x; j) S_g(x; j-1)} \right] h_{j-1}.$$

Substituting the above into (3.12) gives us the differential relations (3.7) and (3.8) completely in terms of the recurrence coefficients and the branch points.

Note that, generally when $\gamma_l(k) \in (\alpha_l, \beta_l)$ $k \geq 1$, we can evaluate $\Psi_n(\alpha_l)$ by taking the limit $x \rightarrow \alpha_l$ in (3.11). From (1.13) and (1.14) we see that for $x = \alpha_l$ the only non-zero term appearing in the sum in (3.12) is the $j = 1$ term. For $j > 1$, $G_{g+1}(\alpha_l; j)$ cannot be zero since this would imply, from (1.13) and (1.14) that $\gamma_l(j) = \alpha_l$. We have,

$$\lim_{x \rightarrow \alpha_l} 2i \Psi_n(x) = \lim_{x \rightarrow \alpha_l} \ln \left(\frac{G_{g+1}(x; 1) + y(x)}{G_{g+1}(x; 1) - y(x)} \right) \pmod{2\pi}.$$

We know that,

$$G_{g+1}(x; 1) = (x - b_1) \prod_{j=1}^g (x - \alpha_j)$$

and therefore,

$$\lim_{x \rightarrow \alpha_l} \Psi_n(x) = \frac{\pi}{2} \pmod{\pi}.$$

3b. Discriminants

In this section we derive an expression for the discriminant when $g = 1$. Once we have knowledge of the differential relations satisfied by a sequence of polynomials one can in general use this information to say something about the discriminant. The discriminant is useful when we consider certain electrostatic problems regarding the zeroes of the polynomials. See [16] and [13] for more details. The expressions we

derive in this section are restricted to the cases when the polynomials do not have zeroes at the branch points. If we write,

$$P_n(x) = \prod_{j=1}^n (x - x_{j,n})$$

then the discriminant is defined to be the following,

$$(3.13) \quad D[P_n(x)] := \prod_{1 \leq i < j \leq n} (x_{i,n} - x_{j,n})^2.$$

Since

$$P'_n(x) = \sum_{j=1}^n \frac{P_n(x)}{(x - x_{j,n})}$$

it is easily verified that,

$$(-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^n P'_n(x_{j,n}) = D[P_n(x)].$$

From (3.7), assuming that $P_n(x)$ has no zeroes at the branch points, we have,

$$P'_n(x_{j,n}) = f_2(x_{j,n}) Q_n(x_{j,n}).$$

To evaluate $Q_n(x_{j,n})$, recall that,

$$P_n(x) = \rho_n(x) \cos \Psi_n(x),$$

and

$$Q_n(x) = i\psi(x)\rho_n(x) \sin \Psi_n(x).$$

Since we have,

$$\frac{1}{i\psi(x)} \frac{Q_n(x)}{P_n(x)} = \tan \Psi_n(x),$$

it must be that,

$$\tan \Psi_n(x_{j,n}) = \pm \infty,$$

and therefore if $x_{j,n}$ is a zero of $P_n(x)$ then we do have,

$$\cos \Psi_n(x_{j,n}) = 0.$$

Hence,

$$\sin \Psi_n(x_{j,n}) = \pm 1.$$

Therefore, it is always true that,

$$|Q_n(x_{j,n})| = |i\psi(x_{j,n})\rho_n(x_{j,n})|.$$

The sign depends on the value of n and j . To show this dependence, we note that the following quantity,

$$i\psi(x_{j,n})\rho_n(x_{j,n}) = i\sqrt{\frac{2h_n}{x_{j,n}^2 - 1} \prod_{l=1}^g \frac{x_{j,n} - \gamma_l(n)}{x_{j,n} - \beta_l}},$$

is positive. Since theorem (7.1) in [9] states that any zero of $P_n(x)$ on the interval (α_l, β_l) must lie in (α_l, γ_l) and since the zeroes of $P_n(x)$ and $Q_n(x)$ interlace, we must have

$$\sin \Psi_{2n}(x_{j,2n}) = (-1)^j$$

and,

$$\sin \Psi_{2n-1}(x_{j,2n-1}) = (-1)^{j+1}.$$

Therefore,

$$Q_n(x_{j,n}) = (-1)^{n+j} i\psi(x_{j,n})\rho_n(x_{j,n}),$$

and,

$$\begin{aligned} D[P_n(x)] &= (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^n P'_n(x_{j,n}) \\ &= (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^n (-1)^{n+j+\frac{1}{2}} \sqrt{2h_n} \sqrt{\frac{\prod_{i=1}^g (x_{j,n} - \gamma_i(n))}{(x_{j,n}^2 - 1) \prod_{i=1}^g (x_{j,n} - \beta_i)}} f_2(x_{j,n}) \\ (3.14) \quad &= (-1)^{\frac{n}{2}} (2h_n)^{\frac{n}{2}} \sqrt{\frac{\prod_{j=1}^g P_n(\gamma_j(n))}{P_n(1)P_n(-1) \prod_{j=1}^g P_n(\beta_j)}} \prod_{j=1}^n f_2(x_{j,n}). \end{aligned}$$

Now in this most general form we can not yet evaluate the discriminant for general g since we do not know how to evaluate the product of the $f_2(x_{j,n})$. However, examining the form of $f_2(x; n)$ we see that it is a rational function. In certain special cases we may know enough about this function to factor it in which case we can express the unknown quantity in (3.17) as $P_n(x)$ evaluated at the zeroes and poles of $f_2(x; n)$. For $g = 1$ we can always do this, in which case,

$$D[P_n(x)] = (-1)^{\frac{n}{2}} (2h_n)^{\frac{n}{2}} \sqrt{\frac{P_n(\gamma(n))}{P_n(1)P_n(-1)P_n(\beta)}} \prod_{j=1}^n f_2(x_{j,n}).$$

Now,

$$f_2(x; n) = \frac{n(x - r_+(n))(x - r_-(n))}{(x - \gamma(n))(x - \alpha)}$$

where,

$$r_{\pm}(n) = \frac{(c_0/n + \gamma) \pm \sqrt{(c_0/n - \gamma)^2 + 2y_1/n}}{2}.$$

Therefore, we have for the square of the discriminant

$$(3.15) \quad D^2[P_n(x)] = (2n^2 h_n)^n \frac{(-1)^n P_n^2(r_+) P_n^2(r_-)}{P_n^2(\alpha) P_n(1) P_n(-1) P_n(\beta) P_n(\gamma(n))}$$

The product $P_n(\beta)P_n(\gamma(n))$ is always positive since we know from [9] that $P_n(x)$ cannot have a zero in the interval $(\gamma(n), \beta)$. A similar calculation for the polynomials of the second kind gives,

$$(3.16) \quad D^2[Q_n(x)] = (2n^2 h_n)^{n-1} \frac{Q_n^2(r_+(n)) Q_n^2(r_-(n))}{Q_n^2(1) Q_n^2(-1) Q_n^2(\beta) Q_n(\gamma(n)) Q_n(\alpha)}.$$

Again, in this case we require that $Q_n(x)$ have no zeroes at the branch points.

These discriminants are valid for genus 1. Indeed we know explicitly the values of $\gamma(n)$ and $r_{\pm}(n)$ for $g = 1$. It is clear from the derivations in this section, that the equivalent expressions for general genus g are,

$$(3.17) \quad D^2[P_n(x)] = (2n^2 h_n)^n \frac{(-1)^n \prod_{j=1}^{2g} P_n^2(r_j(n))}{P_n(1) P_n(-1) \prod_{j=1}^g [P_n(\beta_j) P_n(\gamma_j(n)) P_n^2(\alpha_j)]},$$

and,

$$(3.18) \quad D^2[Q_n(x)] = (2n^2 h_n)^{n-1} \frac{\prod_{j=1}^{2g} Q_n^2(r_j(n))}{Q_n^2(1) Q_n^2(-1) \prod_{j=1}^g [Q_n^2(\beta_j) Q_n(\gamma_j(n)) Q_n(\alpha_j)]},$$

where the $\{r_j(n)\}$ are the $2g$ roots of the polynomial,

$$\left[\prod_{j=1}^g (x - \gamma_j(n)) \right] \left(nx^g - \sum_{j=0}^{g-1} c_j(n) x^j - \frac{1}{2} \sum_{j=1}^g \frac{y_j}{x - \gamma_j(n)} \right).$$

4 Connection with Polynomial mappings

The theta function expressions for the recurrence coefficients a_n and b_n of the generalized Chebyshev polynomials are given in (1.5) and (1.6). Consider for the moment two different values of the recurrence coefficients, for example, b_n and $b_{n+\mathfrak{K}}$, where n and \mathfrak{K} are both integers. If \mathfrak{K} is chosen such that $\mathfrak{K}\hat{\mathbf{B}} \in \mathbb{Z}^g$ then from the periodicity properties of the theta functions we have,

$$b_{n+\mathfrak{K}} = b_n.$$

From the expression for the a_n coefficients, we will also have,

$$a_{n+\mathfrak{K}} = a_n \quad n \geq 2.$$

When the entries of the vector $\hat{\mathbf{B}}$ are all rational numbers, we can always find an integer \mathfrak{K} such that $\mathfrak{K}\hat{\mathbf{B}} \in \mathbb{Z}^g$. If we take the smallest integer that satisfies this requirement we will then have the smallest integer over which the recurrence coefficients themselves repeat, that is the recurrence coefficients are periodic with period \mathfrak{K} . If at least one of the entries is an irrational number we are unable to find an integer \mathfrak{K} that satisfies the periodicity requirement.

We will study the effect that this periodicity has on our previous constructions and use this to explain the connection between the generalized Chebyshev polynomials and non-trivial polynomial mappings [10]. Throughout this section \mathfrak{K} is a given fixed integer. First consider $P_{n\mathfrak{K}}(x)$. From (1.5), and with the periodicity condition, we see that the θ factors cancel in equation (1.5), therefore

$$(4.1) \quad P_{n\mathfrak{K}}(x) = \frac{\Delta_{n\mathfrak{K}}}{2} (e^{n\mathfrak{K}(\Omega(p_x))} + e^{-n\mathfrak{K}(\Omega(p_x))}),$$

hence,

$$(4.2) \quad P_{\mathfrak{K}}(x) = \frac{\Delta_{\mathfrak{K}}}{2} (e^{\mathfrak{K}(\Omega(p_x))} + e^{-\mathfrak{K}(\Omega(p_x))}).$$

Here,

$$(4.3) \quad \Delta_{n\mathfrak{K}} = 2e^{-n\mathfrak{K}\chi_0}.$$

Therefore,

$$(4.4) \quad \begin{aligned} P_{n\mathfrak{K}}(x) &= 2^{n-1} \Delta_{n\mathfrak{K}} T_n \left(\frac{e^{\mathfrak{K}\Omega(x)} + e^{-\mathfrak{K}\Omega(x)}}{2} \right) \\ &= \Delta_{\mathfrak{K}}^n T_n \left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}} \right), \end{aligned}$$

where $T_n(x)$ is the monic Chebyshev polynomial. From the expression for h_n in (1.10) we see that,

$$h_{\mathfrak{K}} = 2e^{-2\mathfrak{K}\chi_0}.$$

It follows that,

$$\Delta_{\mathfrak{K}} = \sqrt{2h_{\mathfrak{K}}},$$

and therefore,

$$(4.5) \quad P_{n\mathfrak{K}}(x) = (2h_{\mathfrak{K}})^{\frac{n}{2}} T_n \left(\frac{P_{\mathfrak{K}}(x)}{\sqrt{2h_{\mathfrak{K}}}} \right).$$

The corresponding formula for the polynomials of the second kind is,

$$(4.6) \quad Q_{n\mathfrak{K}}(x) = (2h_{\mathfrak{K}})^{\frac{n-1}{2}} Q_{\mathfrak{K}}(x) U_n \left(\frac{P_{\mathfrak{K}}(x)}{\sqrt{2h_{\mathfrak{K}}}} \right),$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind (degree $n-1$). Recall that,

$$\mathcal{E}_n(\mathfrak{p}_x) \tilde{\mathcal{E}}_n(\mathfrak{p}_x) = \frac{S_g(x; n)}{\prod_{j=1}^g (x - \alpha_j)}.$$

When $\mathfrak{K}\hat{\mathbf{B}} \in \mathbb{Z}^g$, we have,

$$\mathcal{E}_n(\mathfrak{p}_x)\tilde{\mathcal{E}}_n(\mathfrak{p}_x) = 4e^{-2n\chi_0}.$$

Therefore, since the $\gamma_j(n)$ are the zeroes of $S_g(x; n)$ we must have $\gamma_j(\mathfrak{K}) = \alpha_j$. It will be shown later in this section that the maximum value of g permitted for the polynomials is $\mathfrak{K} - 1$. Therefore, if we put $g = \mathfrak{K} - 1$ and $n = \mathfrak{K}$ in (1.13) we see immediately that,

$$Q_{\mathfrak{K}}(x) = \prod_{j=1}^{\mathfrak{K}-1} (x - \alpha_j).$$

Hence,

$$Q_{n\mathfrak{K}}(x) = (2h_{\mathfrak{K}})^{\frac{n-1}{2}} U_n \left(\frac{P_{\mathfrak{K}}(x)}{\sqrt{2h_{\mathfrak{K}}}} \right) \prod_{j=1}^{\mathfrak{K}-1} (x - \alpha_j).$$

In the language of (3.3), (4.5) becomes,

$$P_{n\mathfrak{K}}(x) = \frac{(\prod_{j=1}^{\mathfrak{K}} f_+(x, j))^n + (\prod_{j=1}^{\mathfrak{K}} f_-(x, j))^n}{2^n},$$

and (4.6) becomes,

$$Q_{n\mathfrak{K}}(x) = \psi(x) \frac{(\prod_{j=1}^{\mathfrak{K}} f_+(x, j))^n - (\prod_{j=1}^{\mathfrak{K}} f_-(x, j))^n}{2^n}.$$

Using the theta function expressions, we can verify the following,

$$f_+(x, \mathfrak{K} + 1) = \frac{\tilde{\mathcal{E}}_{\mathfrak{K}+1}(\mathfrak{p}_x)}{\tilde{\mathcal{E}}_{\mathfrak{K}}(\mathfrak{p}_x)} = \frac{\tilde{\Delta}_{\mathfrak{K}+1}}{\tilde{\Delta}_{\mathfrak{K}}\tilde{\Delta}_1} f_+(x, 1).$$

Similarly we can deduce that,

$$\frac{\tilde{\Delta}_{\mathfrak{K}+1}}{\tilde{\Delta}_{\mathfrak{K}}\tilde{\Delta}_1} = \frac{1}{2},$$

and therefore,

$$f_{\pm}(x, \mathfrak{K} + 1) = \frac{1}{2} f_{\pm}(x, 1).$$

An analogous calculation shows that,

$$f_{\pm}(x, \mathfrak{K} + j) = f_{\pm}(x, j), \quad 1 < j < \mathfrak{K},$$

as we expect from looking at the explicit structure of the $f_{\pm}(x, j)$.

If we know the period \mathfrak{K} then we can use the $P_{\mathfrak{K}}(x)$ with (4.5) to generate every \mathfrak{K}^{th} polynomial in the sequence. To obtain the intermediate polynomials $P_{n\mathfrak{K}+j}$ for $j = 1.. \mathfrak{K} - 1$ we use the fact that

$$\mathcal{E}_{n\mathfrak{K}+j}(\mathfrak{p}_x) = \frac{1}{2} \left(\prod_{l=1}^j f_+(x; l) \right) \mathcal{E}_{n\mathfrak{K}}(\mathfrak{p}_x) = \frac{1}{2} \mathcal{E}_j(\mathfrak{p}_x) \mathcal{E}_{n\mathfrak{K}}(\mathfrak{p}_x),$$

and therefore,

$$(4.7) \quad P_{n\mathfrak{K}+j}(x) = \frac{1}{2} \left(P_j(x)P_{n\mathfrak{K}}(x) + \frac{1}{\psi^2(x)} Q_j(x)Q_{n\mathfrak{K}}(x) \right), \quad 1 \leq j < \mathfrak{K}, \quad n \geq 1$$

Similarly,

$$(4.8) \quad Q_{n\mathfrak{K}+j}(x) = \frac{1}{2} (P_j(x)Q_{n\mathfrak{K}}(x) + Q_j(x)P_{n\mathfrak{K}}(x)), \quad 1 \leq j < \mathfrak{K}, \quad n \geq 1$$

So if we know the first \mathfrak{K} polynomials of the first and second kind we can generate all of the remaining polynomials using (4.5), (4.6), (4.7) and (4.8).

From the theory of the classical Chebyshev polynomials we have,

$$T_m(x)T_n(x) = T_{m+n}(x) + \frac{1}{2^{2n}} T_{m-n}(x), \quad m \geq n \neq 0$$

Similarly,

$$U_m(x)T_n(x) = \begin{cases} U_{m+n}(x) + \frac{1}{2^{2n}} U_{m-n}(x), & m \geq n \neq 0 \\ U_{m+n}(x) - \frac{1}{2^{2m}} U_{n-m}(x), & n \geq m \neq 0 \end{cases}.$$

Analogous formulas to the ones above do not hold for all of the generalized Chebyshev polynomials. However, $P_{n\mathfrak{K}}$ play a similar role to the classical Chebyshev polynomials.

Lemma 1 *For any $m, n \in \mathbb{Z}_+$ we have*

$$(4.9) \quad P_n(x)P_{m\mathfrak{K}}(x) = \begin{cases} P_{n+m\mathfrak{K}}(x) + \frac{h_{m\mathfrak{K}}}{2} P_{n-m\mathfrak{K}}(x) & n \geq m\mathfrak{K} \neq 0 \\ P_{n+m\mathfrak{K}}(x) + \frac{h_n}{2} P_{m\mathfrak{K}-n}(x) & m\mathfrak{K} \geq n \neq 0 \end{cases}$$

Proof :

From section 1,

$$P_n(x) = e^{-n\chi_0} \frac{\vartheta(\mathbf{u}^+)}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}})} \left[e^{n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x - n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} + e^{-n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} \right],$$

and,

$$P_{m\mathfrak{K}}(x) = e^{-m\mathfrak{K}\chi_0} \left[e^{m\mathfrak{K}\Omega(\mathfrak{p}_x)} + e^{-m\mathfrak{K}\Omega(\mathfrak{p}_x)} \right].$$

Therefore,

$$\begin{aligned} P_n(x)P_{m\mathfrak{K}}(x) &= e^{-(n+m\mathfrak{K})\chi_0} \frac{\vartheta(\mathbf{u}^+)}{\vartheta(\mathbf{u}^+ - n\hat{\mathbf{B}})} \left[e^{(n+m\mathfrak{K})\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x - n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} + e^{-(n+m\mathfrak{K})\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} \right. \\ &\quad \left. + e^{(n-m\mathfrak{K})\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x - n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} + e^{-(n-m\mathfrak{K})\Omega(\mathfrak{p}_x)} \frac{\vartheta(\mathbf{u}_x + n\hat{\mathbf{B}})}{\vartheta(\mathbf{u}_x)} \right]. \end{aligned}$$

From the periodicity of the theta functions we may add the vector $\pm m\mathfrak{K}\hat{\mathbf{B}}$ to any of the arguments of the theta functions without changing the equality. We make this change in the argument of the theta functions in such a way that we are left with (4.9). \square

Similarly, for the polynomials of the second kind, we have,

$$Q_n(x)P_{m\mathfrak{K}}(x) = \begin{cases} Q_{n+m\mathfrak{K}}(x) + \frac{h_{m\mathfrak{K}}}{2}Q_{n-m\mathfrak{K}}(x) & n \geq m\mathfrak{K} \neq 0 \\ Q_{n+m\mathfrak{K}}(x) - \frac{h_n}{2}Q_{m\mathfrak{K}-n}(x) & m\mathfrak{K} \geq n \neq 0 \end{cases}.$$

Combining (4.9) and (4.7) we obtain,

$$P_{n\mathfrak{K}-j}(x) = \frac{1}{h_j} \left(P_{nk}(x)P_j(x) - \frac{Q_{nk}(x)Q_j(x)}{\psi^2(x)} \right) \quad j = 1, \dots, \mathfrak{K} - 1.$$

Similarly,

$$Q_{n\mathfrak{K}-j}(x) = \frac{1}{h_j} (Q_{n\mathfrak{K}}(x)P_j(x) - P_{n\mathfrak{K}}(x)Q_j(x)).$$

From (1.8) it follows from the periodicity properties of the theta functions that,

$$a_j = a_{\mathfrak{K}-j+1} \quad j = 1, \dots, \mathfrak{K} - 1.$$

The b_n also have a similar behaviour, and one can show that

$$b_j = b_{\mathfrak{K}-j+2} \quad j = 1, \dots, \mathfrak{K} - 1.$$

However, this is not obvious from the form of the recurrence coefficients given in (1.9). To prove the above equality we start with the three term recurrence relation (1.4) with $n = \mathfrak{K} - j + 1$ and $x = 1$, so that

$$P_{\mathfrak{K}-j+1}(1) = P_{\mathfrak{K}-j+2}(1) + b_{\mathfrak{K}-j+2}P_{\mathfrak{K}-j+1}(1) + a_{\mathfrak{K}-j+1}P_{\mathfrak{K}-j}(1).$$

From the expression for $P_{n\mathfrak{K}-j}(x)$ we have,

$$P_{n\mathfrak{K}-j}(1) = \frac{P_{n\mathfrak{K}}(1)P_j(1)}{h_j}.$$

Substituting this into the three term recurrence relation we obtain,

$$P_{j-1}(1) = P_j(1) + b_{\mathfrak{K}-j+2}P_{j-1}(1) + a_{j-1}P_{j-2}(1).$$

Comparing this with (1.4) at $x = 1$ and $n = j - 1$ we see that $b_{\mathfrak{K}-j+2} = b_j$.

Theorem 4.1 *Given a set E such that the recurrence coefficients have period \mathfrak{K} , then the polynomial $P_{\mathfrak{K}}(x)/\Delta_{\mathfrak{K}}$ maps each sub-interval of E onto $[-1, 1]$.*

Proof: Take an arbitrary polynomial $M_{\mathfrak{K}}(x)$ of degree \mathfrak{K} with leading coefficient $l_{\mathfrak{K}}$ where all of the zeroes lie in $(-1, 1)$. Furthermore,

$$|M_{\mathfrak{K}}(x_j)| \geq 1$$

where the x_j are the stationary points, and,

$$\begin{aligned} M_{\mathfrak{K}}(1) &= 1 \\ M_{\mathfrak{K}}(-1) &= (-1)^{\mathfrak{K}}. \end{aligned}$$

Then the inverse image of $[-1, 1]$ under $M_{\mathfrak{K}}$ is a set J of at most \mathfrak{K} disjoint intervals of the form $J := [-1, \alpha] \cup_{j=1}^{\mathfrak{K}-2} [\beta_j, \alpha_{j+1}] \cup [\beta_{\mathfrak{K}-1}, 1]$, with $-1 \leq \alpha_j \leq \beta_j \leq \alpha_{j+1} \leq 1$. Let $\sigma_T(x)$ be the weight function associated with the Chebyshev polynomials,

$$\sigma_T(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}.$$

Following [10] we define a function $W(x)$ by,

$$(4.10) \quad \frac{1}{\pi} \sqrt{\frac{\prod_{j=1}^{\mathfrak{K}-1} (x - \alpha_j)}{(1-x^2) \prod_{j=1}^{\mathfrak{K}-1} (x - \beta_j)}} = \sigma_A(x) = W(x) \sigma_T(M_{\mathfrak{K}}(x)).$$

Now, $M_{\mathfrak{K}}(x)$ has the property that it maps all of the branch points into -1 or 1 , and therefore,

$$M_{\mathfrak{K}}^2(x) - 1 = l_{\mathfrak{K}}^2 (x^2 - 1) \prod_{j=1}^{\mathfrak{K}-1} (x - \alpha_j)(x - \beta_j).$$

Since $M_{\mathfrak{K}}(x)$ is a polynomial, the above equation imposes constraints on the branch points. Substituting for $M_{\mathfrak{K}}^2(x)$ in (4.10) we have,

$$W(x) = l_{\mathfrak{K}} \prod_{j=1}^{\mathfrak{K}-1} (x - \alpha_j).$$

Now

$$(4.11) \quad \frac{W(z)}{M_{\mathfrak{K}}(z) - x} = \frac{\prod_{i=1}^{\mathfrak{K}-1} (z - \alpha_i)}{\prod_{i=1}^{\mathfrak{K}} (z - M_i^{-1}(x))} = \sum_{i=1}^{\mathfrak{K}} \frac{w_i(x)}{z - M_i^{-1}(x)}$$

where the $M_i^{-1}(x)$ are a complete set of inverse branches of $M_{\mathfrak{K}}(x)$. Now,

$$W(z) = (M_{\mathfrak{K}}(z) - x) \sum_{i=1}^{\mathfrak{K}} \frac{w_i(x)}{z - M_i^{-1}(x)}$$

and

$$W(M_i^{-1}(x)) = w_i(x) \sum_{j \neq i} (M_i^{-1}(x) - M_j^{-1}(x)),$$

and therefore,

$$w_i(x) = \frac{W(M_i^{-1}(x))}{M'(M_i^{-1}(x))}.$$

Also, by comparing coefficients of $z^{\mathfrak{K}-1}$ in (4.11),

$$\sum_{i=1}^{\mathfrak{K}} w_i(x) = 1.$$

From the orthogonality relationship of the Chebyshev polynomials we have,

$$\begin{aligned} h_n^{(T)} \delta_{nm} &= \int_{-1}^1 T_n(z) T_m(z) \sigma_T(z) dz \\ &= \sum_{i=1}^{\mathfrak{K}} \int_{-1}^1 T_n(z) T_m(z) \left(\frac{W(M_i^{-1}(z))}{M'(M_i^{-1}(z))} \right) \sigma_T(z) dz \\ &= \int_J T_n(M_{\mathfrak{K}}(x)) T_m(M_{\mathfrak{K}}(x)) \frac{W(x)}{M'_{\mathfrak{K}}(x)} \sigma_T(M_{\mathfrak{K}}(x)) M'_{\mathfrak{K}}(x) dx \\ &= \int_J T_n(M_{\mathfrak{K}}(x)) T_m(M_{\mathfrak{K}}(x)) \sigma_A(x) dx, \end{aligned}$$

where the last line follows from (4.10). Using the general method in [10] theorem 1, it can be shown that

$$\int_J x^j T_n(M_{\mathfrak{K}}(x)) \sigma_A(x) dx = 0 \quad j < n\mathfrak{K}.$$

Therefore,

$$T_n(M_{\mathfrak{K}}(x)) = c_n P_{n\mathfrak{K}}(x) \quad n \geq 0,$$

where c_n is some constant. So for $n = 1$,

$$M_{\mathfrak{K}}(x) = c_1 P_{\mathfrak{K}}(x).$$

Now, from (4.5),

$$P_{\mathfrak{K}}(1) = \Delta_{\mathfrak{K}}$$

and therefore $c_1 = \frac{1}{\Delta_{\mathfrak{K}}}$, and we have proven that $\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}$ maps J into $[-1, 1]$ \mathfrak{K} -fold. Also, since for the polynomials orthogonal on the set J we have the shown (4.5) to hold, we can conclude that the recurrence coefficients are periodic with period \mathfrak{K} and therefore J is in fact the set E referred to in the statement of the theorem. \square

Remark 1. This theorem has appeared in a more general setting, see [14] and [15] where necessary and sufficient conditions were given for the existence of an extremal polynomial on a certain set E , which corresponds to our mapping polynomial. In general the orthogonal polynomials on these sets have asymptotically periodic recurrence coefficients. These are found to be orthogonal with respect to a more general class of weight functions similar to $w(x)$ of the form,

$$\frac{\pi_{\mathfrak{K}-1}(x)}{y(x)}$$

where $\pi_{\mathfrak{K}-1}(x)$ is a polynomial of degree $\mathfrak{K}-1$ chosen such that the weight function is positive on the interior of the set E . In equation (1.12) of [15] the extremal polynomial of degree n , $\mathcal{T}_n(z)$, is $\frac{P_{\mathfrak{K}}(z)}{\Delta_{\mathfrak{K}}}$ where n , the asymptotic period is identified with our \mathfrak{K} .

As a consequence of this theorem, when the recurrence coefficients have period \mathfrak{K} the set E has at most \mathfrak{K} sub intervals. Recall that g is the number of gaps in $[-1, 1]$. Therefore, the maximum value of g for this set is $\mathfrak{K}-1$.

In figure 1 we show an example of a mapping polynomial for $\mathfrak{K} = 3$.

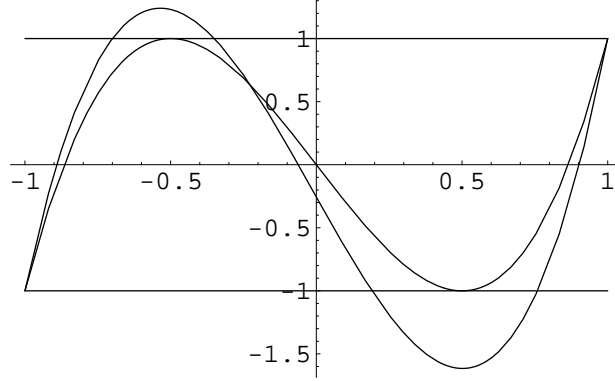


Figure 1: General Form of $M_3(x)$ compared with the Chebyshev polynomial of degree 3. The branch points are where the polynomial crosses the lines $y = -1$ and $y = 1$.

We now consider some examples of sequences of orthogonal polynomials generated via the polynomial mappings $P_{\mathfrak{K}}(x)$. We choose small values of \mathfrak{K} and investigate. $\mathfrak{K} = 1$ corresponds to the Chebyshev polynomials as the recurrence coefficients are constant.

For $\mathfrak{K} = 2$, the mapping polynomial,

$$M_2(x) = \frac{1}{\Delta_2} P_2(x) = \frac{1}{\Delta_2} (x^2 + ax + b_2),$$

satisfies the following conditions

$$\begin{aligned} \frac{1}{\Delta_2} P_2(-1) &= \frac{1}{\Delta_2} P_2(1) = 1 \\ \frac{1}{\Delta_2} P_2(\alpha) &= \frac{1}{\Delta_2} P_2(\beta) = -1. \end{aligned}$$

This implies that $M_2(x)$ has the following form

$$M_2(x) = \frac{1}{\Delta_2} x^2 + \left(1 - \frac{1}{\Delta_2}\right),$$

with

$$\begin{aligned} \alpha^2 &= \beta^2 \\ \Delta_2 &= \frac{1 - \alpha^2}{2} = \frac{1 - \beta^2}{2}. \end{aligned}$$

Therefore,

$$M_2(x) = \frac{2}{1-\alpha^2}x^2 + \left(1 - \frac{2}{1-\alpha^2}\right),$$

with the condition on the branch points, $\alpha = -\beta$. The condition above, tells us which sets E arise from a polynomial mapping of degree 2. These are the same sets that give rise to periodic recurrence coefficients with period 2. To obtain the recurrence coefficients for these polynomials, we can substitute $\beta = -\alpha$ into the general genus 1 recurrence coefficients. The general expressions for the recurrence coefficients can be obtained in one of two ways. Appendix A shows how we can manipulate the theta function expressions for the recurrence coefficients into expressions involving the Jacobian elliptic functions. We are then able to invert the functions and in principle obtain the recurrence coefficients for any n in terms of the branch points only. In this case we only need to find the first two since the others repeat. The second way to find the recurrence coefficients is to use the non-linear difference equations derived in section 2. Once we have the initial conditions we can iterate the non-linear difference equations to find the remaining coefficients. This method is more practical when we are considering higher periodicity. In any case, the recurrence coefficients are found to be,

$$(4.12) \quad \begin{aligned} b_n &= (-1)^n \alpha \\ a_n &= \frac{1}{4}(1-\alpha^2) \quad n \geq 2. \end{aligned}$$

With these coefficients, our polynomials become,

$$P_{2n}(x) = (\Delta_2)^n T_n \left(\frac{P_2(x)}{\Delta_2} \right)$$

and

$$P_{2n+1}(x) = \frac{(x+\alpha)}{2} (\Delta_2)^n \left(T_n \left(\frac{P_2(x)}{\Delta_2} \right) + \frac{(x^2-1)}{\Delta_2} U_n \left(\frac{P_2(x)}{\Delta_2} \right) \right).$$

Also the auxiliary polynomials become,

$$\begin{aligned} S_1(x, n) &= 2h_n(x + (-1)^{n+1}\alpha) \\ G_2(x, n) &= h_{n-1}(x + \alpha)(x - \alpha). \end{aligned}$$

From (3.10) we have,

$$f_1(x; n) = \frac{1}{2} \left(\frac{1}{x + (-1)^{n+1}\alpha} - \frac{1}{x - \alpha} \right).$$

To evaluate $f_2(x; n)$ we must know the $c_0(n)$ and $\gamma(n)$. For genus 1 polynomials we have from [9],

$$c_0(n) = p_1(n) + n \frac{\alpha + \beta}{2} + \frac{\gamma(n) - \alpha}{2}.$$

where $p_1(n)$ is the coefficient of x^{n-1} in $P_n(x)$. In this case,

$$\begin{aligned} p_1(2n) &= 0 \\ p_1(2n-1) &= \alpha. \end{aligned}$$

Therefore,

$$c_0(n) = 0.$$

Since $\gamma(n)$ is the zero of $S_g(x, n)$,

$$\gamma(n) = (-1)^n \alpha,$$

and,

$$f_2(x; n) = \frac{nx}{x - \alpha}.$$

For even n , the discriminant becomes ($n = 2m$),

$$D[P_{2m}(x)] = (-1)^m 2^{-(2m-1)(2m-2)-m} (2m)^{2m} (1 - \alpha^2)^{m(2m-1)} T_m \left(\frac{\alpha^2 + 1}{\alpha^2 - 1} \right)$$

and for odd n , ($n = 2m + 1$)

$$D[P_{2m+1}(x)] = (-1)^m 2^{-4m^2+m-1} (2m+1)^{2m} (1 - \alpha^2)^{m(2m+1)} \left[T_m \left(\frac{\alpha^2 + 1}{\alpha^2 - 1} \right) - \frac{2}{1 - \alpha^2} U_m \left(\frac{\alpha^2 + 1}{\alpha^2 - 1} \right) \right].$$

The discriminant for the case when n is odd cannot be obtained directly from (3.15) since the $P_{2n+1}(x)$ have zeroes at $x = -\alpha$. The problem is that of evaluating $P'_{2n+1}(-\alpha)$. In general evaluating the derivative at one of the branch points is not straightforward. However, in this case it can be verified that,

$$P'_{2n+1}(-\alpha) = 2f_2(-\alpha; 2n+1)Q_{2n+1}(-\alpha).$$

whereas, for the zeroes that do not lie on the branch points we have,

$$P'_{2n+1}(x_{j,2n+1}) = f_2(x_{j,2n+1}; 2n+1)Q_{2n+1}(x_{j,2n+1})$$

and the factor of 2 does not appear. It is for this reason that the expression given in (3.15) remains valid in this case if we insert a factor of 2.

The polynomials in this example can also be shown to satisfy the following differential equation,

$$P''_n - \frac{x^4 - \alpha^2}{x(x^2 - \alpha^2)(1 - x^2)} P'_n + \frac{n^2 x^2}{(1 - x^2)(x^2 - \alpha^2)} P_n = 0 \quad \text{for even } n,$$

and,

$$P''_n - \frac{x^4 - 2\alpha x(1 - x^2) + \alpha^2}{x(x^2 - \alpha^2)(1 - x^2)} P'_n + \frac{n^2 x^4 - 2x\alpha + (n^2 + 1)x^3\alpha - \alpha^2}{x(1 - x^2)(x - \alpha)(x + \alpha)^2} P_n = 0 \quad \text{for odd } n.$$

For the next example we choose $\mathfrak{K} = 3$. Therefore our polynomials can be described in terms of a polynomial mapping of degree 3. The highest genus permitted in this case is 2. This corresponds to three

disjoint intervals. Genus one cases arise if we close one of the gaps. In the last example we saw that the only sets that gave rise to a period of 2 were symmetric around the origin, that is $\alpha = -\beta$. The aim of this example is to determine which sets give rise to recurrence coefficients with period 3. We determine the recurrence coefficients, the γ points and under a certain condition, the $c_j(n)$.

We have a mapping polynomial $M_3(x)$ with leading coefficient $\frac{1}{\Delta_3}$ that satisfies the following equations,

$$\begin{aligned} M_3(\pm 1) &= \pm 1 \\ M_3(\alpha_1) &= M_3(\beta_1) = 1 \\ M_3(\alpha_2) &= M_3(\beta_2) = -1. \end{aligned}$$

Solving these six equations we have,

$$M_3(x) = \frac{x^3}{\Delta_3} + ax^2 + \left(1 - \frac{1}{\Delta_3}\right)x - a,$$

with,

$$(4.13) \quad a = \frac{-(\beta_1 + \alpha_1 + 1)}{(\beta_1 + 1)(\alpha_1 + 1)} = \frac{-(\alpha_2 + \beta_2 - 1)}{(\beta_2 - 1)(\alpha_2 - 1)},$$

and,

$$(4.14) \quad \Delta_3 = (\beta_1 + 1)(\alpha_1 + 1) = (\beta_2 - 1)(\alpha_2 - 1).$$

Since we can characterize the mapping polynomial in terms of two branch points, we have a system with two unknowns. We describe the system in terms of α_1 and β_1 . The form of the mapping polynomial restricts α_1 and β_1 to certain intervals on $[-1, 1]$. If we eliminate α_2 from (4.13) and (4.14) we get,

$$\beta_2 = \frac{\alpha_1 + \beta_1 + 2 \pm \sqrt{(\beta_1 - \alpha_1)^2 - 4(\alpha_1 + \beta_1 + 1)}}{2}$$

hence,

$$(4.15) \quad (\alpha_1 - \beta_1)^2 - 4(\alpha_1 + \beta_1 + 1) \geq 0.$$

Therefore we have,

$$\alpha_1 \leq \beta_1 + 2 - 4\sqrt{\frac{\beta_1 + 1}{2}}.$$

This inequality, coupled with the inequality,

$$\alpha_1 \leq \beta_1$$

implies that α_1 lies on the interval $[-1, -\frac{1}{2}]$. Therefore β_1 lies on the interval $[\alpha_1, \beta_{1,max}]$ where,

$$\beta_{1,max} = 2 + \alpha_1 - 4\sqrt{\frac{\alpha_1 + 1}{2}}.$$

The value of $\beta_{1,max}$ follows from (4.15) with the equality sign. With these conditions on α_1 and β_1 we have,

$$(4.16) \quad \alpha_2 = 1 + \frac{\alpha_1 + \beta_1}{2} - \frac{1}{2} \sqrt{(\beta_1 - \alpha_1)^2 - 4(\alpha_1 + \beta_1 + 1)}$$

and,

$$(4.17) \quad \beta_2 = 1 + \frac{\alpha_1 + \beta_1}{2} + \frac{1}{2} \sqrt{(\beta_1 - \alpha_1)^2 - 4(\alpha_1 + \beta_1 + 1)}.$$

For $g = 2$ we evaluate the first three recurrence coefficients in the most general case in terms of α_1 , α_2 , β_1 and β_2 , using (1.13) and (1.14). Then we substitute for α_2 and β_2 using (4.16) and (4.17) and find,

$$(4.18) \quad \begin{aligned} a_1 &= \frac{1}{2}(1 + \alpha_1) \left(2 - \alpha_1 + \beta_1 + \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right) \\ a_2 &= \frac{1}{16} \left(\alpha_1 - \beta_1 - 2 + \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right)^2 \\ a_3 &= \frac{1}{4}(1 + \alpha_1) \left(2 - \alpha_1 + \beta_1 + \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right), \end{aligned}$$

and,

$$(4.19) \quad \begin{aligned} b_1 &= \frac{1}{2} \left(\beta_1 - \alpha_1 + \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right) \\ b_2 &= \frac{1}{4} \left(2 + 3\alpha_1 + \beta_1 - \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right) \\ b_3 &= \frac{1}{4} \left(2 + 3\alpha_1 + \beta_1 - \sqrt{(\beta_1 - \alpha_1)^2 - 4(1 + \alpha_1 + \beta_1)} \right). \end{aligned}$$

We also find that, for $n \geq 2$,

$$\begin{aligned} \gamma_{1,2}(n) &= \frac{1}{2} \left(1 + \alpha_1 + \beta_1 - b_{n+1} \right. \\ &\quad \left. \mp \sqrt{1 - 2(\alpha_1 + \beta_1) + (\alpha_1 - \beta_1)^2 - 4(a_n + a_{n+1}) + 2b_{n+1}(1 + \alpha_1 + \beta_1 - \frac{3}{2}b_{n+1})} \right). \end{aligned}$$

Expressions for the polynomials are easily obtained from our equations in the previous sections. Expressions for the discriminant and the differential equation can in principle be obtained using the constants that we have derived in this section.

A highly symmetric case corresponding to $g = 2$ arises when $\beta_1 + \alpha_1 = -1$. With this, the a defined by (4.13) is 0, which implies that

$$\alpha_2 + \beta_2 = 1.$$

Substituting these into (4.14), gives,

$$\beta_2 = 1 + \beta_1.$$

This case corresponds to the first gap being symmetric about the point $-\frac{1}{2}$ and the second gap being symmetric about the point $\frac{1}{2}$. In this case the square roots appearing in the recurrence coefficients vanish leaving (with $\alpha = \alpha_1$),

$$\begin{aligned} a_1 &= -2\alpha(1 + \alpha) \\ a_2 &= \frac{1}{4} \\ a_3 &= -\alpha(1 + \alpha) \end{aligned}$$

and

$$\begin{aligned} b_1 &= -(1 + 2\alpha) \\ b_2 &= \frac{1}{2} + \alpha \\ b_3 &= \frac{1}{2} + \alpha \end{aligned}$$

Also,

$$G_3(x, n) = h_{n-1}(x^3 + (2a_n - 1 - \alpha - \alpha^2)x + 2a_n(b_n + b_{n+1}))$$

and

$$S_2(x, n) = 2h_n(x^2 + b_{n+1}x + a_n + a_{n+1} + b_{n+1}^2 - 1 - \alpha - \alpha^2).$$

This gives

$$\gamma_{1,2}(n) = \frac{-b_{n+1} \mp \sqrt{b_{n+1}^2 - 4(a_n + a_{n+1} + b_{n+1}^2 - 1 - \alpha - \alpha^2)}}{2}.$$

The first three polynomials are,

$$\begin{aligned} P_1(x) &= x + 2\alpha + 1 \\ P_2(x) &= x^2 + (\alpha + \frac{1}{2})x - \frac{1}{2} \\ P_3(x) &= x^3 - (1 + \alpha + \alpha^2)x. \end{aligned}$$

From (3.9), (3.10) and (3.12) we can calculate $f_1(x; n)$ and $f_2(x; n)$.

Returning to the previous non-symmetric genus 2 example, we add the restriction that one of the gaps is closed. We take the case where $\alpha_2 = \beta_2$. In this situation (4.13) and (4.14) simplify to

$$a = \frac{-(\beta_1 + \alpha_1 + 1)}{(\beta_1 + 1)(\alpha_1 + 1)} = \frac{-(2\alpha_2 - 1)}{(\alpha_2 - 1)^2},$$

and

$$\Delta_3 = (\beta_1 + 1)(\alpha_1 + 1) = (\alpha_2 - 1)^2.$$

Therefore, eliminating α_2 from the above we obtain,

$$(\beta_1 + 1)(\alpha_1 + 1) = \left(\frac{\beta_1 + \alpha_1}{2} \right)^2,$$

with the solutions,

$$\beta_1 = \alpha_1 + 2 \pm 4\sqrt{\frac{\alpha_1 + 1}{2}}.$$

Since $\alpha_1 \leq \beta_1$ we must have

$$(4.20) \quad \beta_1 = \alpha_1 + 2 - 4\sqrt{\frac{\alpha_1 + 1}{2}}.$$

This relation governs the variation of β_1 as α_1 varies over the interval $[-1, -\frac{1}{2}]$. It describes a contour in the (α_1, β_1) plane. Figure (2) shows a graphical representation of this contour with similar curves for higher values of \mathfrak{R} . Had we have chosen the condition $\alpha_1 = \beta_1$ for the other $g = 1$ case we would have obtained,

$$(4.21) \quad \beta_2 = -2 + \alpha_2 + 4\sqrt{\frac{1 - \alpha_2}{2}},$$

where α_2 varies over $[-1, \frac{1}{2}]$. Using the method outlined in the appendix we are able to compute the recurrence coefficients for $g = 1$ explicitly. From now on we refer to α_1 and β_1 as just α and β respectively. Upon substituting for β given by (4.21) into the general recurrence coefficients we obtain the following,

$$\begin{aligned} a_1 &= 2(\alpha + 1) \left(1 - \sqrt{\frac{1 + \alpha}{2}} \right) \\ a_2 &= \frac{\alpha + 3}{2} - 2\sqrt{\frac{\alpha + 1}{2}} \\ a_3 &= (\alpha + 1) \left(1 - \sqrt{\frac{1 + \alpha}{2}} \right) \end{aligned}$$

and,

$$\begin{aligned} b_1 &= 1 - 2\sqrt{\frac{1 + \alpha}{2}} \\ b_2 &= (\alpha + 1) - \sqrt{\frac{1 + \alpha}{2}} \\ b_3 &= (\alpha + 1) - \sqrt{\frac{1 + \alpha}{2}}. \end{aligned}$$

The recurrence coefficients repeat in blocks of three however we must remember that $a_{3n+1} = \frac{a_1}{2}$, $n \geq 1$. Furthermore,

$$\begin{aligned} c_0(3n) &= n \left(\alpha - 2\sqrt{\frac{1 + \alpha}{2}} \right) \\ c_0(3n + 1) &= \frac{1}{2} \left((1 + 2n)\alpha - (1 + 4n)\sqrt{\frac{1 + \alpha}{2}} \right) \\ c_0(3n + 2) &= \frac{1}{2} \left((1 + 2n)\alpha - (3 + 4n)\sqrt{\frac{1 + \alpha}{2}} \right), \end{aligned}$$

and,

$$\begin{aligned}\gamma(3n) &= \alpha \\ \gamma(3n+1) &= -\sqrt{\frac{1+\alpha}{2}} \\ \gamma(3n+2) &= -\sqrt{\frac{1+\alpha}{2}}.\end{aligned}$$

These constants, listed above, allow one to calculate the differential equations and the discriminant given in the previous section.

General \mathfrak{K}

For notational convenience we now label as $E_{\mathfrak{K}}$ the sets E that give rise to period \mathfrak{K} recurrence coefficients. Equivalently these are the sets for which the minimum degree of polynomial mapping from $E_{\mathfrak{K}}$ onto $[-1, 1]$ is \mathfrak{K} . From our earlier examples,

$$\begin{aligned}E_1 &= [-1, 1] \\ E_2 &= [-1, \alpha] \cup [-\alpha, 1] \quad \alpha \in [-1, 0] \\ E_3 &= [-1, \alpha] \cup [\beta, u(\alpha, \beta)] \cup [v(\alpha, \beta), 1] \quad \alpha \in [-1, -\frac{1}{2}] \quad \beta \in [\alpha, \beta_{max}],\end{aligned}$$

where,

$$\begin{aligned}u(\alpha, \beta) &= 1 + \frac{\alpha + \beta}{2} - \frac{1}{2}\sqrt{(\beta - \alpha)^2 - 4(\alpha + \beta + 1)} \\ v(\alpha, \beta) &= 1 + \frac{\alpha + \beta}{2} + \frac{1}{2}\sqrt{(\beta - \alpha)^2 - 4(\alpha + \beta + 1)}.\end{aligned}$$

Each set $E_{\mathfrak{K}}$ is parametrized in terms of $\mathfrak{K} - 1$ variables. As \mathfrak{K} increases it becomes intractable to solve explicitly for the relationships between the branch points. However, the branch points are constrained on certain curves in $[-1, 1]^{2g}$ where the co-ordinate axes of the $2g$ cube label the position of the branch points.

In what follows, we describe the constraints on the mapping polynomial. This will provide a set of equations that define the curves of constant periodicity in the $2g$ cube. We also show how to determine the smallest \mathfrak{K} for a given configuration of the branch points.

For general \mathfrak{K} , the $2\mathfrak{K}$ conditions imposed on the mapping polynomial,

$$M_{\mathfrak{K}}(x) = a_{\mathfrak{K}}x^{\mathfrak{K}} + a_{\mathfrak{K}-1}x^{\mathfrak{K}-1} + \dots + a_1x + a_0,$$

are

$$\begin{aligned}M_{\mathfrak{K}}(-1) &= (-1)^{\mathfrak{K}} \\ M_{\mathfrak{K}}(1) &= 1 \\ (4.22) \quad M_{\mathfrak{K}}(\alpha_j) &= M_{\mathfrak{K}}(\beta_j) = (-1)^{\mathfrak{K}+j} \quad j = 1, \dots, \mathfrak{K} - 1.\end{aligned}$$

Therefore $\mathfrak{K} + 1$ of these will determine the coefficients in terms of $\mathfrak{K} + 1$ branch points. Using the first two equations of (4.22) and any $\mathfrak{K} - 1$ of the remaining ones we can solve for the coefficients of the mapping

polynomial in terms of 1, -1 and $\mathfrak{K} - 1$ free parameters. The other $\mathfrak{K} - 1$ branch points are related to the free parameters by the remaining equations of (4.22). These equations determine a surface in the $2g$ cube. The surface obtained in this way will correspond to maximum $g = \mathfrak{K} - 1$. Closing one of the gaps, that is setting $\alpha_j = \beta_j$ for some $j \leq \mathfrak{K} - 1$, will determine surfaces corresponding to that value of \mathfrak{K} for smaller values of g .

If we close all of the gaps by setting $\alpha_j = \beta_j$ for all $j = 1 \dots \mathfrak{K} - 1$, then the mapping polynomial becomes,

$$M_{\mathfrak{K}}(x) = 2^{\mathfrak{K}-1} T_{\mathfrak{K}}(x).$$

If we denote by \tilde{x}_j the point where $\alpha_j = \beta_j$, then we must have

$$2^{\mathfrak{K}-1} T_{\mathfrak{K}}(\tilde{x}_j) = \pm 1$$

from the properties of the mapping polynomial. Since $T_n(x) = 2^{1-n} \cos(n\theta)$, $x = \cos(\theta)$ we see that the set of points $\{\tilde{x}_j\}_{j=1}^{\mathfrak{K}-1}$ are the $\mathfrak{K} - 1$ stationary points of the Chebyshev polynomial of degree \mathfrak{K} . Therefore, each gap i.e. $(\alpha_i, \beta_i)_{i=1}^{\mathfrak{K}-1}$ can be said to be 'centred' (not necessarily symmetrically) around one of the stationary points of the Chebyshev polynomial of degree \mathfrak{K} . The most simple example of this is the $\mathfrak{K} = 2$ case where the gap is centred around the point $x = 0$. In the $\mathfrak{K} = 3$, $g = 2$ case, the two gaps are centred around $x = \frac{1}{2}$ and $x = -\frac{1}{2}$. The stationary points of $T_4(x)$ are $x = -\frac{1}{\sqrt{2}}$, $x = 0$, and $x = \frac{1}{\sqrt{2}}$. So for a polynomial mapping of degree 4 we have three distinct $g = 1$ cases corresponding to one gap which is opened around one of the stationary points.

Consider now the $\mathfrak{K} = 2$ mapping polynomial, $\frac{P_2(x)}{\Delta_2}$, which maps $[-1, \alpha] \cup [-\alpha, 1]$ to $[-1, 1]$. From (4.4),

$$P_4(x) = \Delta_2^2 T_2\left(\frac{P_2(x)}{\Delta_2}\right).$$

Since, $\Delta_2^2 = 2\Delta_4$, and from the fact that the classical Chebyshev polynomials map $[-1, 1]$ to $[-1, 1]$ we see that, $\frac{P_4(x)}{\Delta_4}$ also maps $[-1, \alpha] \cup [-\alpha, 1]$ to $[-1, 1]$. Therefore, it is this configuration of the branch points which corresponds to the $\mathfrak{K} = 4$, $g = 1$ case where the gap is centered around $x = 0$. Indeed for any mapping polynomial of even degree, one of the $g = 1$ cases corresponds to this configuration, since a Chebyshev polynomial of even degree has a stationary point at zero.

The same reasoning as above applies for general \mathfrak{K} . that is, if $\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}$ is a mapping polynomial for a certain configuration of the branch points, then so are the polynomials $\frac{P_{n\mathfrak{K}}(x)}{\Delta_{n\mathfrak{K}}}$. Therefore amongst all of the configurations that correspond to a mapping of degree $n\mathfrak{K}$, there can be found those that correspond to degree $j\mathfrak{K}$, where j divides n .

If \tilde{x}_l is a stationary point of $\hat{T}_{\mathfrak{K}}(x) := 2^{\mathfrak{K}-1} T_{\mathfrak{K}}(x)$, then

$$|\hat{T}_{\mathfrak{K}}(\tilde{x}_l)| = 1$$

and

$$|\hat{T}_{j\mathfrak{K}}(\tilde{x}_l)| = 1,$$

which indicates that \tilde{x}_l is also a stationary point of $\hat{T}_{j\mathfrak{K}}(x)$. For example $\hat{T}_{12}(x)$ has amongst its stationary points, the stationary points of $\hat{T}_6(x)$, $\hat{T}_4(x)$, $\hat{T}_3(x)$ and $\hat{T}_2(x)$.

So as an example, a mapping polynomial of degree 12 will give rise to eleven $g = 1$, single parameter cases of the generalized Chebyshev polynomials. Six of these will already be the same as those that arise from a mapping of degree 6 and four will be the same as those that arise from a mapping of degree 4. All of these cases cover those that arise from mappings of degree 2 and 3. Therefore a mapping of degree 12 will give rise to two $g = 1$ cases that cannot be generated from polynomial mappings of lower degree.

The upshot of the above discussion is that we can uniquely label each surface in the $2g$ cube by a set of points corresponding to the stationary points of the Chebyshev polynomials of any degree. We do this in the following way.

If we denote by s_l the ordered set of all stationary points of the Chebyshev polynomial of degree l , then we can label a surface by taking any subset of the s_l . The number of elements in a subset is the number of gaps g in the interval $[-1, 1]$. For example,

$$\begin{aligned} s_2 &= \{0\} \\ s_3 &= \left\{-\frac{1}{2}, \frac{1}{2}\right\} \\ s_4 &= \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\} \\ s_5 &= \left\{-\frac{1}{4}(\sqrt{5}+1), -\frac{1}{4}(\sqrt{5}-1), \frac{1}{4}(\sqrt{5}-1), \frac{1}{4}(\sqrt{5}+1)\right\}. \end{aligned}$$

Therefore, the set,

$$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

corresponds to the $\mathfrak{K} = 4$ $g = 2$ surface where both gaps are centred around the two points $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$. This is a two parameter curve. Those curves shown in figure 2 can be labelled by one number which is a stationary point of a Chebyshev polynomial. It is clear from the discussion so far that the number \mathcal{N} given by,

$$\mathcal{N} = \binom{\mathfrak{K}-1}{g},$$

is the number of surfaces in the $2g$ cube associated with the integer \mathfrak{K} .

So far, we have discussed in general terms how to describe the surfaces corresponding to a particular value of \mathfrak{K} . On the other hand, the problem of finding \mathfrak{K} from a given set E appears to be a very difficult numerical one. The condition for periodicity of the recurrence coefficients is

$$(4.23) \quad \mathfrak{K}\hat{\mathbf{B}} \in \mathbb{Z}^g \setminus \{0^g\}.$$

where,

$$\hat{\mathbf{B}}_j = \frac{1}{2\pi i} \int_{b_j} d\Omega, \quad j = 1, \dots, g.$$

If $-\hat{\mathbf{B}}_j$ are rational then \mathfrak{K} is the least common multiple of the denominators of these rational numbers. For example, if

$$\hat{\mathbf{B}} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{5} \\ -\frac{1}{6} \end{pmatrix}$$

then $\mathfrak{K} = 30$. However, if at least one of the components of the vector $\hat{\mathbf{B}}$ is an irrational number then it is impossible to find an integer \mathfrak{K} that satisfies condition (4.25). These correspond to the cases where the recurrence coefficients never repeat themselves or equivalently, the sets for which there is no polynomial mapping from this set onto $[-1, 1]$.

Note that in general the values of $\hat{\mathbf{B}}_j$ are subject to certain conditions. In [9], section 10, a physical interpretation was given; the values $-\hat{\mathbf{B}}_j$ were seen to be the proportion of the overall charge lying on

the interval $[\beta_{j-1}, \alpha_j]$, where $\beta_0 := -1$, in the equilibrium charge density for the set E , when the total charge was normalized to 1, that is,

$$\int_E \sigma(x) dx = 1.$$

Since the total charge on all of the intervals must add up to 1 the following inequality is satisfied by the components of $\hat{\mathbf{B}}$,

$$0 < \sum_{j=1}^g -\hat{\mathbf{B}}_j < 1.$$

To finish this section we refer to figure 2. Recall that each point in the plane represents a configuration of the branch points for $g = 1$ and therefore a sequence of orthogonal polynomials. Using our notation, if $\{\tilde{x}_l\}$ represents a curve in this plane with integer \mathfrak{K} then $\{-x_i\}$ represents a different curve but with the same integer \mathfrak{K} . We ask the question “Is there a relationship between the polynomials associated with the curves $\{x_i\}$ and $\{-x_i\}$?” A similar question can also be asked for $g > 1$. The relation between these cases is given in the most general form by theorem 1.2. However, since the recurrence coefficients are periodic, the elements appearing in the determinant simplify:

$$P_{n\mathfrak{K}+j}(\beta_l) = \frac{1}{2} P_{n\mathfrak{K}}(\beta_l) P_j(\beta_l) \quad 1 \leq j \leq \mathfrak{K} - 1$$

and

$$Q_{n\mathfrak{K}+j}(\alpha_l) = \frac{1}{2} P_{n\mathfrak{K}}(\alpha_l) Q_j(\alpha_l) \quad 1 \leq j \leq \mathfrak{K} - 1$$

where,

$$P_{n\mathfrak{K}}(\alpha_l) = P_{n\mathfrak{K}}(\beta_l) = \frac{\Delta_{\mathfrak{K}}^n (-1)^{n(l+\mathfrak{K})}}{2^{n-1}}.$$

As an example we show the relation between the polynomials associated with the curves $\{-\frac{1}{2}\}$ and $\{\frac{1}{2}\}$ which are the two $\mathfrak{K} = 3$ $g = 1$ curves. Using the notation of theorem 1.2, we denote the polynomials associated

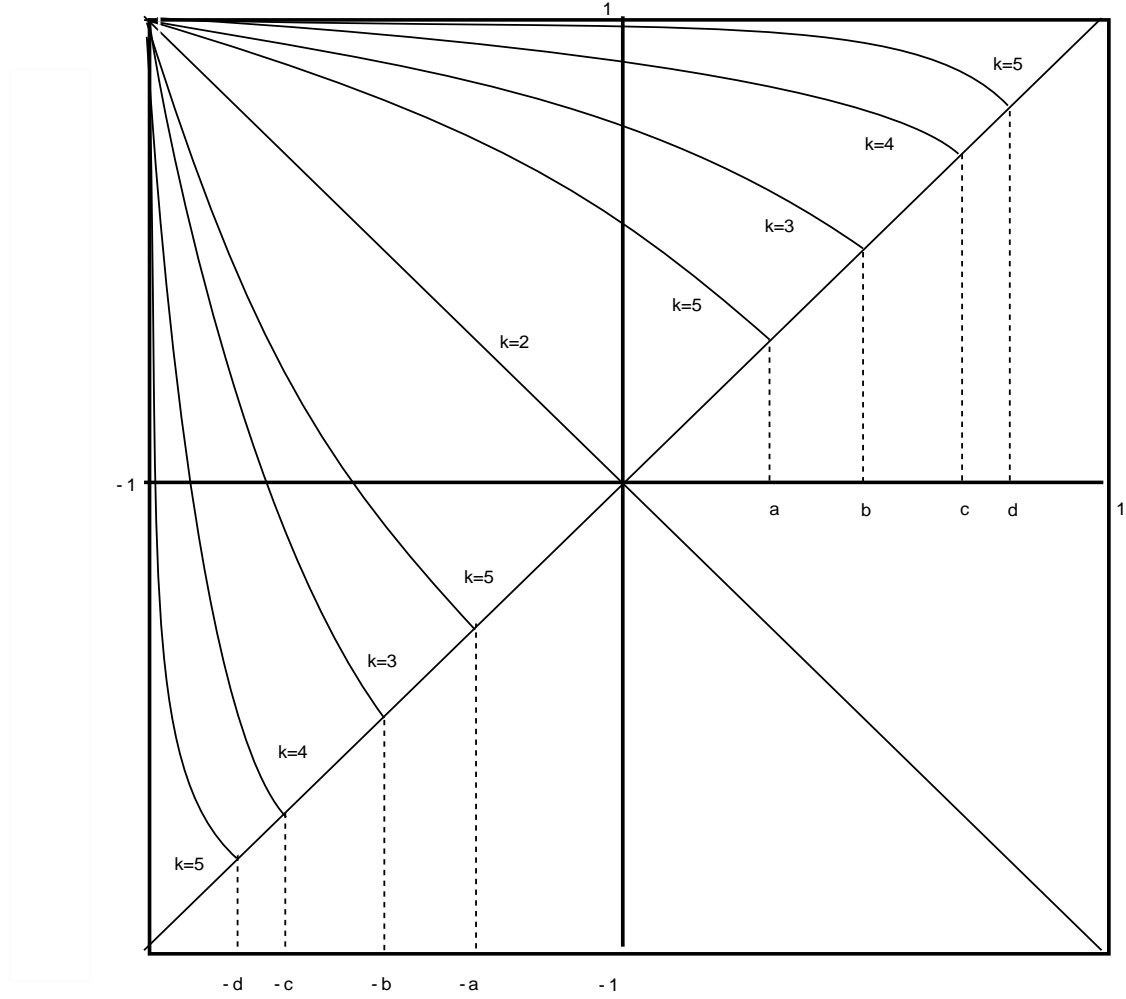


Figure 2: Contours of constant periodicity in the (α, β) plane for genus 1. Here $a = (-1 + \sqrt{5})/4$, $b = 1/2$, $c = 1/\sqrt{2}$ and $d = (1 + \sqrt{5})/4$. This is not a precise plot, it is representative of the relative position of the contours.

with the set $\{-\frac{1}{2}\}$ as $P_n(x)$, the polynomials associated with the set $\{\frac{1}{2}\}$ as $\tilde{P}_n(x)$, and the respective mapping polynomials as $M_3(x)$ and $\tilde{M}_3(x)$. It is easy to verify that,

$$M_3(x) = (-1)\tilde{M}_3(-x).$$

Consequently,

$$\tilde{P}_{3n}(x) = (-1)^{3n}P_{3n}(-x).$$

However, for the intermediate polynomials we must use the expressions given in theorem 1.2, which give,

$$\begin{aligned} \tilde{P}_{3n+1}(x) = & \frac{(-1)^{3n}}{\left(x + \alpha + 2 - 4\sqrt{\frac{\alpha+1}{2}}\right)} \left[P_{3n+2}(-x) - \left(\sqrt{\frac{\alpha+1}{2}} - 1\right) P_{3n+1}(-x) \right. \\ & \left. + \left((\alpha+3)\sqrt{\frac{\alpha+1}{2}} - 2(\alpha+1)\right) P_{3n}(-x) \right], \end{aligned}$$

and,

$$\begin{aligned} \tilde{P}_{3n+2}(x) = & \frac{(-1)^{3n+1}}{\left(x + \alpha + 2 - 4\sqrt{\frac{\alpha+1}{2}}\right)} \left[P_{3(n+1)}(-x) + \left(\sqrt{\frac{\alpha+1}{2}}\right) P_{3n+2}(-x) \right. \\ & \left. - \frac{1}{2} \left(1 + \alpha - 2\sqrt{\frac{1+\alpha}{2}}\right) P_{3n+1}(-x) \right]. \end{aligned}$$

4b. General form of the polynomials

In this section we show some plots of the polynomials for small \mathfrak{K} and prove a theorem regarding the zeroes. The Chebyshev polynomials $\hat{T}_n(x)$ is bounded between 1 and -1 . These are the mapping polynomials from $[-1, 1]$ to $[-1, 1]$. As a consequence of (4.5) we see that $|\hat{P}_{n,\mathfrak{K}}(x)| \leq 1$ for $x \in E$, where,

$$\hat{P}_{n,\mathfrak{K}}(x) = \frac{2^{n-1}}{(\Delta_{\mathfrak{K}})^n} P_{n,\mathfrak{K}}(x).$$

Outside of the set we no longer have this bound. Therefore we would expect the polynomials to appear graphically as in figure 3. In figures 4, 5 and 6 we plot some of the intermediate polynomials $\hat{P}_{n,\mathfrak{K}+j}(x)$ for various values of j and \mathfrak{K} , where,

$$\hat{P}_{n,\mathfrak{K}+j}(x) = \frac{2^n}{(\Delta_{\mathfrak{K}})^n \sqrt{2h_j}} P_{n,\mathfrak{K}+j}(x).$$

In E they are bounded above and below by some function. It is clear from the representation (1.18) that the form of the envelope is $\pm \hat{\rho}_{n,\mathfrak{K}+j}(x)$, for $x \in E$ where,

$$\begin{aligned} \hat{\rho}_{n,\mathfrak{K}+j}(x) &= \frac{2^n}{(\Delta_{\mathfrak{K}})^n \sqrt{2h_j}} \sqrt{\frac{S_g(x, n\mathfrak{K} + j)}{\prod_{l=1}^g (x - \alpha_l)}}, \\ &= \sqrt{\frac{\prod_{j=1}^g (x - \gamma_j(n\mathfrak{K} + j))}{\prod_{j=1}^g (x - \alpha_j)}}. \end{aligned}$$

Note that the quantity under the square root is strictly positive for $x \in E$.

We already know something about the zeros of the generalized chebyshev polynomials in the most general case from [9]. In the polynomial mapping case we can say a little more about the zeros.

Theorem 4.2 *All of the zeros of $P_{n,\mathfrak{K}}(x)$ lie inside E , and in each of the \mathfrak{K} sub-intervals that make up E lie n zeros.*

Proof: All of the zeros of the Chebyshev polynomials are contained in $(-1, 1)$. From (4.5), and since $\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}} \in [-1, 1]$ if and only if $x \in E$, we see that all of the zeros of $P_{n,\mathfrak{K}}(x)$ lie inside E and each subinterval E_i contains n zeros.

The zeros of $Q_{n,\mathfrak{K}}(x)$ behave similarly except in this case there are $n - 1$ zeros in each interval E_i and there are also zeros at every α_j , $j = 1, \dots, \mathfrak{K} - 1$. This is easily verified by checking the form of (4.6). From (4.6) the $n - 1$ zeros in each E_i come for the zeros of $U_n\left(\frac{P_{\mathfrak{K}}}{\Delta_{\mathfrak{K}}}\right)$. In each interval E_i , these zeros interlace with the zeros of $P_{n,\mathfrak{K}}(x)$.

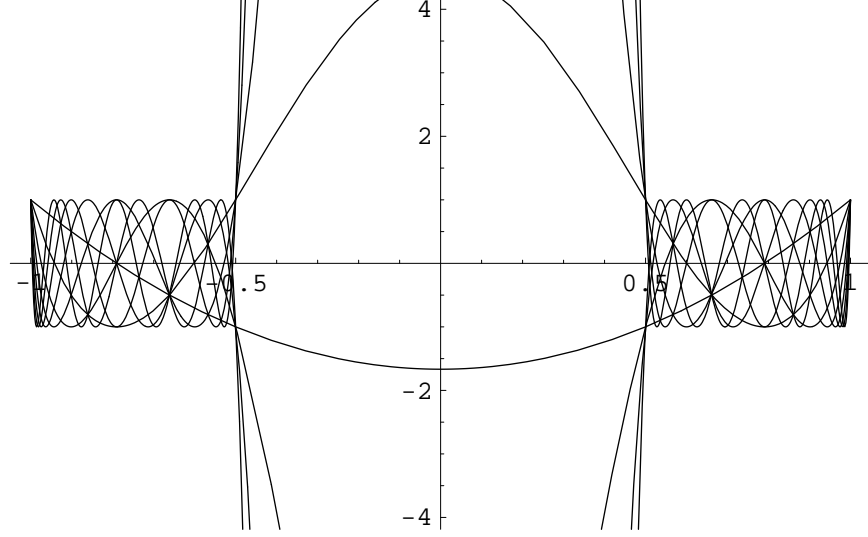


Figure 3: $\mathfrak{K}=2$, $j=0$ $n=1..8$, $\alpha = -0.5$

Theorem 4.3 Let $\{x_{i,n}\}$ denote the zeros of $P_n(x)$ and $\{y_{i,n}\}$ denote the zeros of $Q_n(x)$. Consider the following two sets,

$$\begin{aligned} A &:= \{\{x_{i,n\mathfrak{K}}\}, \{x_{i,j}\}\} \\ B &:= \{-1, 1, \beta_1, \dots, \beta_{\mathfrak{K}-1}, \{y_{i,j}\}, \{y_{i,n\mathfrak{K}}\}\} \setminus \{\alpha_1, \dots, \alpha_{\mathfrak{K}-1}\}. \end{aligned}$$

The following three statements are true :

Between, any zero of $P_j(x)$, and any element of B which is larger than this zero, there lies at least one zero of $P_{n\mathfrak{K}+j}(x)$.

Between any zero of $Q_j(x)$, and any element of A which is larger than this zero there lies at least one zero of $P_{n\mathfrak{K}+j}(x)$.

If $P_j(x)$ has a zero at any of the points in B then $P_{n\mathfrak{K}+j}(x)$ has a zero at the same point.

Proof: From (4.7), $x_{i,n\mathfrak{K}+j}$ solves the following equation,

$$P_{n\mathfrak{K}}(x)P_j(x) = -\frac{1}{\psi^2(x)}Q_{n\mathfrak{K}}(x)Q_j(x).$$

Using (4.5) and (4.6), we see that this is equivalent to,

$$(4.24) \quad \Delta_{\mathfrak{K}}T_n\left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}\right)P_j(x) = U_n\left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}\right)Q_j(x)(1-x^2)\prod_{i=1}^{\mathfrak{K}-1}(x-\beta_i).$$

For clarity, we illustrate a proof of the case of $\mathfrak{K} = 3$ and $n = 2$. The zeroes of $C(x)$ and $D(x)$ where,

$$C(x) := \Delta_{\mathfrak{K}}T_n\left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}\right)$$

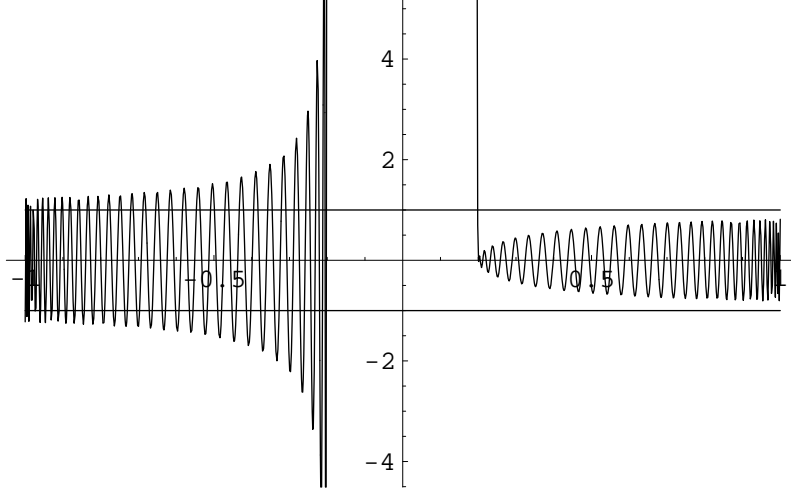


Figure 4: $\mathfrak{K}=2, j=1, n=65, \alpha = -0.2$

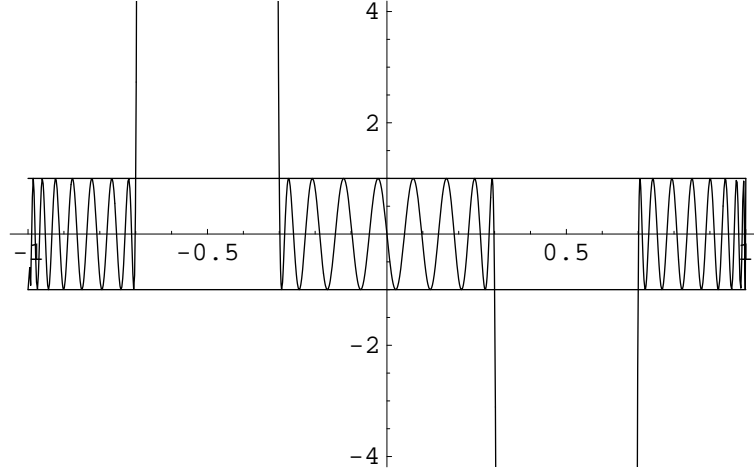


Figure 5: The symmetric case for $\mathfrak{K}=3, j=0, n=17, \alpha = -0.7$

and,

$$D(x) := (1 - x^2)U_n \left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}} \right) \prod_{i=1}^{\mathfrak{K}-1} (x - \beta_i)$$

are indicated in figure 7. The crosses denote the zeros of $C(x)$ and the dots denote the zeros of $D(x)$. We see that they interlace. In order to plot the remaining zeros of (4.24) we must add j crosses and $j - 1$ dots, corresponding to the zeros of $P_j(x)$ and $Q_j(x)$ respectively. We can distribute these without any restriction other than they must also interlace with each other. The result is a sequence of crosses and dots that alternate except for j occurrences of two adjacent crosses and $j - 1$ occurrences of two adjacent dots. Where we see two adjacent crosses, one of them is a zero of $P_j(x)$. Similarly with two dots, one of them is a zero of $Q_j(x)$. As an example we plot where the zeroes of both sides of (4.24) might lie, for $\mathfrak{K} = 3, n = 2$ and $j = 2$ in figure 8.

In order to graph the polynomials that pass through the zeros we need an initial condition. The condition

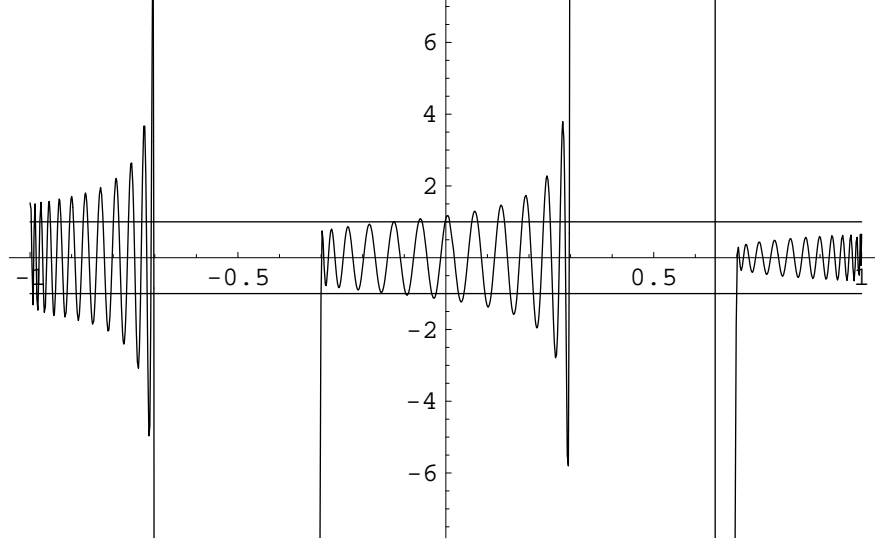


Figure 6: The symmetric case for $\mathfrak{K}=3$, $j=1$, $n=25$, $\alpha = -0.7$



Figure 7: The zeroes of $C(x)$ and $D(x)$ for $\mathfrak{K} = 3$ and $n = 2$. The three line segments are the disjoint intervals that make up E . The crosses denote the zeroes of $C(x)$ and the dots, the zeroes of $D(x)$.

is that both sides of (4.24) have the same sign at $x = -1 + \epsilon$ where ϵ is a small positive constant. This is true because close to $x = -1$, $P_j(x)$ and $Q_j(x)$ must have opposite sign. Similarly, $T_n\left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}\right)$ and $U_n\left(\frac{P_{\mathfrak{K}}(x)}{\Delta_{\mathfrak{K}}}\right)$ have the same sign if \mathfrak{K} is even and have opposite sign if \mathfrak{K} is odd. The product $\prod_{j=1}^{\mathfrak{K}-1} (x - \beta_i)$, for $x = -1 + \epsilon$ is negative if \mathfrak{K} is even and positive if \mathfrak{K} is odd. $\Delta_{\mathfrak{K}}$ is a positive constant, and $(1 - x^2)$ is also positive. Therefore both sides of (4.24) have the same sign at $x = -1 + \epsilon$.

Therefore if we draw lines connecting the dots and the crosses that represent the tonicity of the polynomials we will see the pattern shown in figure 9.

We are interested in where these polynomials intersect, as these points represent the zeros of $P_{n,\mathfrak{K}+j}(x)$. Starting from the point $x = -1$ and moving along the interval $[-1, 1]$, the following behaviour is observed. Before the first occurrence of two adjacent crosses the polynomials intersect between every dot-cross pair. After the appearance of the two adjacent crosses, the polynomials intersect between every cross-dot pair. This continues until we reach the first occurrence of two adjacent dots and then the intersection of the polynomials appears again in between every dot-cross pair. So every appearance of two crosses or two dots swaps the position of the intersection points between dot-cross pairs and cross-dot pairs. Since the appearance of two dots and two crosses alternate, after every appearance of two crosses there must be a point of intersection before the next dot. This corresponds to the statement in the theorem that between every zero of $P_j(x)$ and

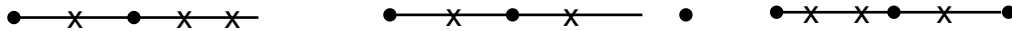


Figure 8:

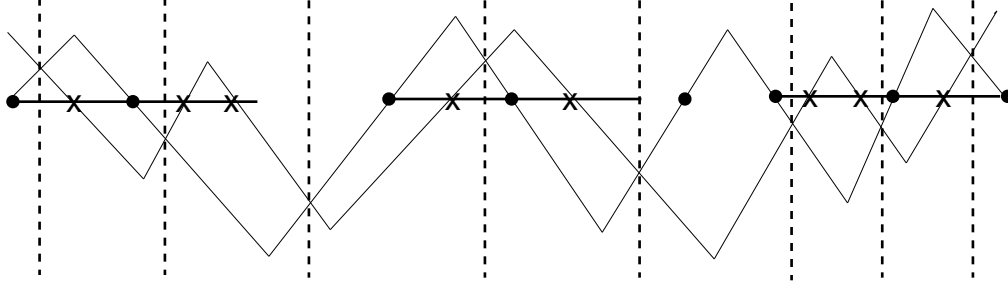


Figure 9: The intersections of the two lines represent qualitatively where the zeroes of $P_{n,\bar{\kappa}+j}(x)$ will lie relative to the zeroes of both sides of (4.24)

the next element from the set B lies a zero of $P_{n,\bar{\kappa}+j}(x)$. Similarly, after the appearance of two dots there must be an intersection point before the next cross. This is equivalent to saying that between every zero of $Q_j(x)$ and the next element from set A also lies a zero of $P_{n,\bar{\kappa}+j}(x)$. \square

Corollary *If $P_j(x)$ has a zero in \bar{E}_i then so does $P_{n,\bar{\kappa}+j}(x)$. ($\bar{E}_i := (\alpha_i, \beta_i)$)*

Proof: Referring to fig 9 we see that there is always a dot at the beta points. If there is a zero of $P_j(x)$ in \bar{E}_i then the previous theorem tells us that there must be a zero of $P_{n,\bar{\kappa}+j}$ between this zero and the next β point. Since \bar{E}_i ends at the next β point the zero of $P_{n,\bar{\kappa}+j}$ is also contained in \bar{E}_i . \square

Appendix A

Genus one recurrence coefficients

Here we show one method of determining the recurrence coefficients for the genus 1 case explicitly in terms of the branch points α and β . It involves manipulating the theta function expressions given in (1.8) and (1.9). We see explicitly how the Jacobian elliptic functions define the form of the coefficients. From [9], for $g = 1$ we have,

$$b_n = \frac{\beta - \alpha}{2} + \frac{1}{A} \left[2 \frac{\vartheta'_3(u^+)}{\vartheta_3(u^+)} + \frac{\vartheta'_3((2n-3)u^+)}{\vartheta_3((2n-3)u^+)} - \frac{\vartheta'_3((2n-1)u^+)}{\vartheta_3((2n-1)u^+)} \right],$$

where

$$A = \frac{4K}{\sqrt{(1-\alpha)(1+\beta)}}$$

and,

$$u^+ = \frac{1}{2K} \int_0^{\sqrt{\frac{\beta+1}{2}}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

with

$$k^2 = \frac{2(\beta - \alpha)}{(1 - \alpha)(1 + \beta)}.$$

Using,

$$\vartheta_3(v) = \vartheta_0(v \pm \frac{1}{2})$$

and,

$$\theta_j(v) = \vartheta_j\left(\frac{v}{2K}\right)$$

we write,

$$b_n = \frac{\beta - \alpha}{2} + \frac{2K}{A} \left[2 \frac{\theta'_0(2Ku^+ + K)}{\theta_0(2Ku^+ + K)} + \frac{\theta'_0((2n-3)2Ku^+ - K)}{\theta_0((2n-3)2Ku^+ - K)} - \frac{\theta'_0((2n-1)2Ku^+ + K)}{\theta_0((2n-1)2Ku^+ + K)} \right].$$

The Jacobi function is defined to be

$$Z(w) = \frac{\theta'_0(w)}{\theta_0(w)},$$

and it satisfies the following formula,

$$Z(u+v) - Z(u-v) - 2Z(v) = \frac{-2k^2 \text{sn}^2(u) \text{sn}(v) \text{cn}(v) \text{dn}(v)}{1 - k^2 \text{sn}^2 u \text{sn}^2 v}.$$

If we make the choice,

$$\begin{aligned} u &= (2n-2)2Ku^+ \\ v &= 2Ku^+ + K, \end{aligned}$$

then we have,

$$\begin{aligned} b_n &= \frac{2(\beta - \alpha)}{\sqrt{(1 - \alpha)(1 + \beta)}} \left[\frac{\text{sn}^2((2n-2)2Ku^+) \text{sn}(2Ku^+ + K) \text{cn}(2Ku^+ + K) \text{dn}(2Ku^+ + K)}{1 - k^2 \text{sn}^2((2n-2)2Ku^+) \text{sn}^2(2Ku^+ + K)} \right] \\ &+ \frac{\beta - \alpha}{2}. \end{aligned}$$

Applying some basic identities ($k' = \sqrt{1 - k^2}$),

$$\begin{aligned} \text{sn}(w + K) &= \frac{\text{cn}(w)}{\text{dn}(w)} \\ \text{cn}(w + K) &= -k' \frac{\text{sn}(w)}{\text{dn}(w)} \\ \text{dn}(w + K) &= \frac{-k'}{\text{dn}(w)} \end{aligned}$$

and,

$$\begin{aligned} \text{sn}^2 w + \text{cn}^2 w &= 1 \\ k^2 \text{sn}^2 w + \text{dn}^2 w &= 1 \end{aligned}$$

we can write,

$$b_n = (\beta - \alpha) \left[\frac{1}{2} - \left(\frac{1 + \alpha}{(\beta - \alpha) - (1 + \beta)/\text{sn}^2((2n-2)2Ku^+)} \right) \right]$$

since,

$$\text{sn}(2Ku^+) = \sqrt{\frac{\beta + 1}{2}}.$$

We have the following addition formula for $\text{sn}(u)$ valid for any u and v ,

$$\text{sn}(u + v) = \frac{\text{sn}(u) \text{cn}(v) \text{dn}(v) + \text{sn}(v) \text{cn}(u) \text{dn}(u)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)}.$$

We can iterate this to obtain $\text{sn}((2n-2)2Ku^+)$ in terms of k and $\text{sn}(2Ku^+)$. This gives us in principle, a method for evaluating the recurrence coefficients in terms of α and β . Owing to the nature of the addition formula, after just a few iterations the expressions for the coefficients become very large.

In [9], the a_n are given as,

$$a_n = \left[\frac{\vartheta_1'(0)}{A\vartheta_1(2u^+)} \right]^2 \frac{\vartheta_3((2n+1)u^+) \vartheta_3((2n-3)u^+)}{\vartheta_3^2((2n-1)u^+)} \quad n \geq 2.$$

Using the following identity,

$$1 - k^2 \text{sn}^2(u) \text{sn}^2(v) = \theta_0(0)^2 \frac{\theta_0(u+v) \theta_0(u-v)}{\theta_0^2(u) \theta_0^2(v)},$$

with

$$\begin{aligned} u &= (2n-1)2Ku^+ + K \\ v &= 4Ku^+ \end{aligned}$$

we have,

$$\begin{aligned} a_n &= c \left(1 - k^2 \text{sn}^2((2n-1)2Ku^+ + K) \text{sn}^2(4Ku^+) \right) \quad n \geq 2. \\ &= c \left(1 - \frac{8(\beta - \alpha)}{(2 - \alpha + \beta)^2} \text{sn}^2((2n-1)2Ku^+ + K) \right). \end{aligned}$$

Here c is a constant independent of n ,

$$c = \left(\frac{\vartheta_1'(0) \vartheta_0(2u^+)}{A \vartheta_0(0) \vartheta_1(2u^+)} \right)^2.$$

To determine c explicitly, we evaluate a_2 by comparing coefficients of (1.13) for $n = 2$ and then compare this with a_2 given above. In which case we find,

$$c = \frac{1}{16} (2 - \alpha + \beta)^2.$$

This value for c was also determined in [9] pg. 4695. Note that in the higher genus cases, the non linear difference equations derived in section 2 can be used to evaluate the recurrence coefficients.

Appendix B

g=2 Auxiliary polynomials

Using the method outlined in section 2 for $g = 2$ we find,

$$\begin{aligned}
\frac{G_3(x, n)}{h_{n-1}} &= x^3 - \frac{1}{2}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)x^2 \\
&\quad - \left(\frac{1}{2} + \frac{(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2)}{8} \right. \\
&\quad \left. - \frac{1}{4}(\beta_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2 + \alpha_1\alpha_2 + \alpha_1\beta_1 + \alpha_1\beta_2) - 2a_n \right) x \\
&\quad + \frac{1}{16} \left(-(\alpha_1^3 + \alpha_2^3 + \beta_1^3 + \beta_2^3) + 4(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \right. \\
&\quad + \beta_1\beta_2(\beta_1 + \beta_2) + \alpha_2\beta_1(\alpha_2 + \beta_1) + \alpha_2\beta_2(\alpha_2 + \beta_2) + \alpha_1\alpha_2(\alpha_1 + \alpha_2) \\
&\quad + \alpha_1\beta_1(\alpha_1 + \beta_1) + \alpha_1\beta_2(\alpha_1 + \beta_2) \\
&\quad - 2(\alpha_2\beta_1\beta_2 + \alpha_1\beta_1\beta_2 + \alpha_1\alpha_2\beta_1 + \alpha_1\alpha_2\beta_2) \\
&\quad \left. - 16a_n(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 2(b_n + b_{n+1})) \right), \quad n \geq 2
\end{aligned}$$

and,

$$\begin{aligned}
\frac{S_2(x, n)}{h_n} &= 2x^2 - [(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - 2b_{n+1}]x \\
&\quad - \frac{1}{4} \left(4 + \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 - 2(\alpha_1\alpha_2 + \alpha_1\beta_1 + \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_2 + \beta_1\beta_2) \right. \\
&\quad \left. - 8(a_{n+1} + a_n) + 4b_{n+1}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - 8b_{n+1}^2 \right), \quad n \geq 1.
\end{aligned}$$

$$G_3(x; 1) = (x - b_1)(x - \alpha_1)(x - \alpha_2)$$

and,

$$S_2(x; 0) = (x - \alpha_1)(x - \alpha_2).$$

Incidentally, this gives us the $\gamma_{1,2}(n)$ coefficients for the genus 2 case as,

$$\begin{aligned}
\gamma_{1,2}(n) &= \frac{1}{4}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - 2b_{n+1} \\
&\quad \mp \left(8 + 3(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) - 2(\alpha_1\alpha_2 + \alpha_1\beta_1 + \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_2 + \beta_1\beta_2) \right. \\
&\quad \left. - 16(a_n + a_{n+1}) + 4b_{n+1}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - 12b_{n+1}^2 \right)^{\frac{1}{2}}, \quad n \geq 1.
\end{aligned}$$

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